

Almost sure multifractal spectrum of SLE

Ewain Gwynne Jason Miller Xin Sun

Massachusetts Institute of Technology

March 8, 2016

Abstract

Suppose that η is a Schramm-Loewner evolution (SLE_κ) in a smoothly bounded simply connected domain $D \subset \mathbf{C}$ and that ϕ is a conformal map from \mathbf{D} to a connected component of $D \setminus \eta([0, t])$ for some $t > 0$. The multifractal spectrum of η is the function $(-1, 1) \rightarrow [0, \infty)$ which, for each $s \in (-1, 1)$, gives the Hausdorff dimension of the set of points $x \in \partial\mathbf{D}$ such that $|\phi'((1 - \epsilon)x)| = \epsilon^{-s+o(1)}$ as $\epsilon \rightarrow 0$. We rigorously compute the a.s. multifractal spectrum of SLE, confirming a prediction due to Duplantier. As corollaries, we confirm a conjecture made by Beliaev and Smirnov for the a.s. bulk integral means spectrum of SLE and we obtain a new derivation of the a.s. Hausdorff dimension of the SLE curve for $\kappa \leq 4$. Our results also hold for the $\text{SLE}_\kappa(\rho)$ processes with general vectors of weight ρ .

Contents

1	Introduction	2
1.1	Multifractal spectrum definition	3
1.2	Main results	4
1.3	Integral means spectrum	6
1.4	Outline	8
2	Preliminaries	10
2.1	Basic notations	10
2.2	Reverse continuity conditions	11
2.3	Schramm-Loewner evolution	14
2.4	Gaussian free fields	15
2.5	Imaginary geometry	17
2.6	Properties of the multifractal spectrum sets	18
2.7	Zero-one laws	19
3	One point estimates for the inverse maps	21
3.1	Reverse SLE martingales and upper bound	22
3.2	Reduction of the lower bound to a result for a stopping time	23
3.3	Derivative estimate via reverse SLE/GFF coupling	24
3.4	Proof of Proposition 3.7	26
3.5	Estimates for chordal SLE in the disk	26
4	One point estimates for the forward maps	27
4.1	Statement of the estimates	27
4.2	Area estimates and stopping estimates for finite time maps	28
4.3	Comparison lemmas	30
4.4	Proof of Theorem 4.1	34
4.5	Finite time estimates	36

5	Upper bounds for multifractal and integral means spectra	36
5.1	Upper bound for the Hausdorff dimension of the subset of the circle	36
5.2	Upper bound for the Hausdorff dimension of the subset of the curve	38
5.3	Upper bound for the integral means spectrum	40
6	Two point estimate	41
6.1	Event at the hitting time	41
6.2	Events for the perfect points	48
6.3	k-perfect points	53
6.4	Analytic properties	55
6.5	Probabilistic properties	59
7	Lower bounds for multifractal and integral means spectra	66
7.1	Setup	66
7.2	Lower bound for the Hausdorff dimension of the subset of the curve	67
7.3	Lower bound for the Hausdorff dimension of the subset of the circle	69
7.4	Proof of Theorem 1.1	71
7.5	Lower bound for the integral means spectrum	71
A	Proof of Proposition 3.10	72
A.1	Pushing the force point to the imaginary axis	73
A.2	Pushing the force point starting from the imaginary axis	75
A.3	Conclusion of the proof	78
B	Comparisons of derivatives using harmonic measure	78
C	Strict mutual absolute continuity for SLE	82

1 Introduction

The Schramm-Loewner evolution (SLE_κ) is a one-parameter family of random fractal curves in a simply connected domain in \mathbf{C} , indexed by $\kappa > 0$. SLE was introduced by Schramm in [Sch00], and has since become a central object of study in both probability theory and statistical physics. See e.g. [Wer04, Law05] for an introduction to SLE. Its importance is that it describes the scaling limit of the interfaces which arise in a number of discrete models in statistical physics, see, e.g., [LSW04, Smi10, SS05, SS09, Mil10].

Roughly speaking, the multifractal spectrum of a domain $D \subset \mathbf{C}$ refers to one of the two functions

$$s \mapsto \dim_{\mathcal{H}} \Theta^s(D) \quad \text{or} \quad s \mapsto \dim_{\mathcal{H}} \tilde{\Theta}^s(D)$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension and $\tilde{\Theta}^s(D)$ is the set of points $x \in \partial D$ with the property that the modulus of the derivative $|\phi'((1 - \epsilon)x)|$ of a conformal map ϕ from the unit disk \mathbf{D} into D grows like ϵ^{-s} as $\epsilon \rightarrow 0$ and $\Theta^s(D) = \phi(\tilde{\Theta}^s(D))$. There are several more or less equivalent definitions of this concept. See Section 1.1 for the precise definition we use in this paper.

The multifractal spectrum of D is a means of quantifying the behavior of $|\phi'|$ near ∂D , even though ϕ need not be differentiable on ∂D . It is closely related to various other quantities associated with ∂D , e.g. the Hausdorff dimension, Hölder regularity, and packing dimension of ∂D ; the integral means spectrum of D ; and the harmonic measure spectrum of the complement of a hull. See [Mak98] for some results in this direction. Such complex analytic quantities are often difficult if not impossible to compute explicitly for specific deterministic domains. However, for random domains (like the complement of an SLE curve) explicit calculations can sometimes be more tractable.

There has been substantial interest in the multifractal properties of SLE_κ (i.e. that of the domain obtained by excising the curve) in both mathematics and physics recent years. For example, it is shown by Beffara in [Bef08] that the a.s. Hausdorff dimension of the SLE_κ curve is $1 + \kappa/8$ for $\kappa \in (0, 8)$ and 2 for $\kappa \geq 8$. The optimal Hölder exponent for the SLE_κ curve is derived in [JVL11], building on the work of Rohde and Schramm [RS05] and Lind [Lin08].

There have also been a number of works which study various versions of the multifractal spectrum of SLE. The first such works [Dup99a, Dup99b], due to Duplantier, give non-rigorous predictions of the multifractal exponents for Brownian motion and self-avoiding random walk, which correspond to SLE_κ for $\kappa = 6$ and $\kappa = 8/3$, respectively. In [Dup00], Duplantier extends this to a non-rigorous prediction of the multifractal spectrum of the SLE_κ curve for general values of $\kappa > 0$. Observing that the predicted multifractal spectrum for SLE_κ in [Dup00] is invariant under the replacement $\kappa \mapsto 16/\kappa$ is what originally led Duplantier to conjecture *SLE duality* (c.f. [Dup00, Dup03]), which states the outer boundary of an SLE_κ curve for $\kappa > 4$ is described by a type of $\text{SLE}_{16/\kappa}$ curve. Various forms of SLE duality have since been rigorously proven in [Zha08a, Zha10, Dub09a, MS16a, MS13].

In [DB02, DB08], the authors study (non-rigorously) a notion of spectrum involving the argument, rather than just the modulus, of the derivative of the SLE maps. In [Dup03], these predictions are expanded to higher multifractal spectra, e.g. the dimension of the set of points on the curve where the behavior of the derivative on *both* sides of the curve is prescribed. See also [Dup04] for additional discussion of these and other multifractal-type spectra.

The first mathematical work on the multifractal spectrum of SLE is due to Beliaev and Smirnov [BS09] in which they compute the average integral means spectrum for a whole-plane SLE curve. Expanding on the results of [BS09], the authors of [DNNZ12] (see also [LY13, LY14]) use exact solutions of differential equations for the moments of the derivatives of the whole-plane SLE maps to study the integral means spectrum of certain SLE and generalized SLE processes. The paper [DHBZ15] extends these calculations to the case of mixed moments for the modulus of an SLE_κ Loewner map and the modulus of its derivative, and studies a generalized integral means spectrum. In [JVL12], the authors rigorously compute the multifractal spectrum at the tip of the SLE curve; this is the first work in which an almost sure result for the multifractal spectrum for SLE is obtained. The authors of [ABJ15] compute the almost sure dimension of the set of points where an SLE_κ curve ($\kappa > 4$) intersects the boundary at a given “angle”. Binder and Duplantier have informed the authors in private communication [BD14] of a forthcoming work in which they prove formulae for the average mixed integral means spectra (i.e. β -spectrum with complex exponent) both in the bulk and at the tip, for chordal SLE. The corresponding formulae agree after Legendre transform with the predictions from [DB02, DB08] concerning the mixed multifractal spectra for harmonic measure and rotation (equivalently, modulus and argument).

In this article, we will give the first rigorous derivation of the a.s. bulk multifractal spectrum of chordal SLE_κ (i.e. that of the complementary domain). We will also obtain the a.s. bulk integral means spectrum of SLE; the spectrum that we find confirms [BS09, Conjecture 1]. Our approach differs from those used elsewhere in the literature to prove results of this type in that we make use of various couplings of SLE processes with the Gaussian free field (GFF). In the proof of the upper bound we use a coupling of the reverse SLE Loewner flow with a free boundary GFF (sometimes called the “quantum zipper”) [She16, MS16d, DMS14]. Our proof of the lower bound will make extensive use of the coupling of SLE with a GFF with Dirichlet boundary conditions (sometimes called the “imaginary geometry” coupling) [She05, Dub09b, MS16a, MS16b, MS16c, MS13]. This latter coupling has also been used to aid in proving lower bounds for the Hausdorff dimensions of sets associated with SLE in [MW14]. Our approach at a high level is similar in spirit to the one used in [MW14], but the technical details are rather different.

Acknowledgments The authors thank Dapeng Zhan for comments on an earlier version of this paper. EG was supported by the Department of Defense via an NDSEG fellowship. JM was partially supported by DMS-1204894. XS was partially supported by DMS-1209044.

1.1 Multifractal spectrum definition

We will now introduce the sets whose Hausdorff dimension we will compute, in the setting of general domains in the complex plane. Our definitions are similar to those in [JVL12, Section 2], but we deal with the boundary of a domain rather than the tip of a given curve.

Let $D \subset \mathbf{C}$ be a simply connected domain and let $\phi : \mathbf{D} \rightarrow D$ be a conformal map. For $s \in \mathbf{R}$, define

$$\tilde{\Theta}^s(D) := \left\{ x \in \partial\mathbf{D} : \lim_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)x)|}{-\log \epsilon} = s \right\} \quad (1.1)$$

and

$$\Theta^s(D) := \phi(\tilde{\Theta}^s(D)). \quad (1.2)$$

Also define

$$\begin{aligned} \tilde{\Theta}^{s;\leq}(D) &:= \left\{ x \in \partial \mathbf{D} : \limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)x)|}{-\log \epsilon} \leq s \right\} \\ \Theta^{s;\leq}(D) &:= \phi(\tilde{\Theta}^{s;\leq}(D)) \\ \tilde{\Theta}^{s;\geq}(D) &:= \left\{ x \in \partial \mathbf{D} : \limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)x)|}{-\log \epsilon} \geq s \right\} \\ \Theta^{s;\geq}(D) &:= \phi(\tilde{\Theta}^{s;\geq}(D)). \end{aligned}$$

The *multifractal spectrum* of D can be defined as one of the two functions $s \mapsto \dim_{\mathcal{H}} \Theta^s(D)$ or $s \mapsto \dim_{\mathcal{H}} \tilde{\Theta}^s(D)$. It is easy to check that these definitions do not depend on the choice of conformal map ϕ . We note that although the sets $\Theta^s(D)$ and $\tilde{\Theta}^s(D)$ are defined for all $s \in \mathbf{R}$, these sets are empty for $s \notin [-1, 1]$ (see Lemma 2.11 below).

1.2 Main results

Our main result is the following theorem.

Theorem 1.1. *Let $\kappa \leq 4$. Let η be a chordal SLE_{κ} from $-i$ to i in \mathbf{D} . Let D_{η} be the connected component of $\mathbf{D} \setminus \eta([0, \infty))$ containing 1 on its boundary. Let*

$$\tilde{\xi}(s) := 1 - \frac{(4+\kappa)^2 s^2}{8\kappa(1+s)} \quad (1.3)$$

$$\xi(s) := \frac{8\kappa(1+s) - (4+\kappa)^2 s^2}{8\kappa(1-s^2)} \quad (1.4)$$

$$s_- := \frac{4\kappa - 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2} \quad (1.5)$$

$$s_+ := \frac{4\kappa + 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2}. \quad (1.6)$$

For $s \in (-1, 1)$, a.s.

$$\begin{aligned} \dim_{\mathcal{H}} \tilde{\Theta}^s(D_{\eta}) &= \dim_{\mathcal{H}} \tilde{\Theta}^{s;\geq}(D_{\eta}) = \tilde{\xi}(s), & 0 \leq s \leq s_+ \\ \dim_{\mathcal{H}} \tilde{\Theta}^s(D_{\eta}) &= \dim_{\mathcal{H}} \tilde{\Theta}^{s;\leq}(D_{\eta}) = \tilde{\xi}(s), & s_- \leq s \leq 0 \\ \dim_{\mathcal{H}} \Theta^s(D_{\eta}) &= \dim_{\mathcal{H}} \Theta^{s;\geq}(D_{\eta}) = \xi(s), & \frac{\kappa}{4} \leq s \leq s_+ \\ \dim_{\mathcal{H}} \Theta^s(D_{\eta}) &= \dim_{\mathcal{H}} \Theta^{s;\leq}(D_{\eta}) = \xi(s), & s_- \leq s \leq \frac{\kappa}{4}. \end{aligned}$$

Moreover, we a.s. have $\tilde{\Theta}^s(D_{\eta}) = \Theta^s(D_{\eta}) = \emptyset$ for each $s \notin [s_-, s_+]$.

Remark 1.2. The significance of s_- and s_+ is that $\tilde{\xi}(s) \geq 0$ for $s \in [s_-, s_+]$, and the significance of $s = \kappa/4$ is that it is the value which maximizes ξ . Note $s_- \in (-1, 0)$ and $s_+ \in (0, 1]$ for any $\kappa > 0$ and $s_+ = 1$ if and only if $\kappa = 4$. We refer the reader to Remark 7.7 below for more detail regarding the case $\kappa = 4$, $s = 1$.

The $\text{SLE}_{\kappa}(\underline{\rho})$ processes are an important variant of SLE in which one keeps track of extra marked points — so-called force points. The force points can be either on the domain boundary or in its interior and are respectively referred to as boundary and interior force points. These processes were first introduced by Lawler, Schramm, and Werner in [LSW03, Section 8.3] and, just like ordinary SLE_{κ} , the $\text{SLE}_{\kappa}(\underline{\rho})$ processes naturally arise in many different contexts. Since $\text{SLE}_{\kappa}(\underline{\rho})$ for different vectors of weights $\underline{\rho}$ has the same

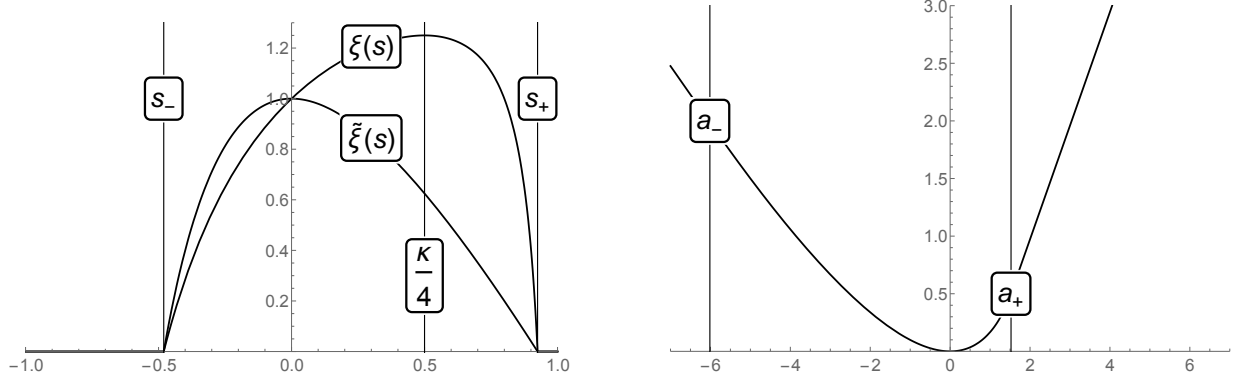


Figure 1.1: Left: A graph of the Hausdorff dimensions $\tilde{\xi}(s)$ of $\tilde{\Theta}^s(D_\eta)$ and $\xi(s)$ of $\Theta^s(D_\eta)$ from Theorem 1.1 as s ranges from -1 to 1 for $\kappa = 2$. The value of s which maximizes $\tilde{\xi}$ is 0 and the value of s which maximizes ξ is $\kappa/4 = 1/2$. Note that $\xi(\kappa/4) = 1 + \kappa/8$ which is the almost sure Hausdorff dimension of SLE_κ [Bef08]. Right: a graph of the bulk integral means spectrum $\text{IMS}_{D_\eta}(a)$ of D_η from Corollary 1.8 as a ranges from -7 to 7 for $\kappa = 3$.

behavior when it is not interacting with its force points, one expects an analogue of Theorem 1.1 to be true for such processes provided we exclude points near the boundary of the domain and stop the path before interacting with an interior force point. Furthermore, by SLE duality, one expects an analogue of Theorem 1.1 for $\kappa > 4$. Such results do indeed hold true, as described in the following corollary.

Corollary 1.3. *Let $D \subset \mathbf{C}$ be a smoothly bounded domain. Let $\kappa > 0$ and let $\underline{\rho}$ be a vector of real weights. Let η be a chordal $\text{SLE}_\kappa(\underline{\rho})$ process in D , with any choice of initial and target points and force points located anywhere in \overline{D} , run up until the first time it either hits an interior force point or hits the continuation threshold (c.f. [MS16a, Section 2.1]). Fix $s \in (-1, 1)$. Almost surely, the following is true. Let V be a connected component of $D \setminus \eta$ or a connected component of $D \setminus \eta([0, t])$ for any $t > 0$ before η hits an interior force point or the continuation threshold and let $\phi : \mathbf{D} \rightarrow V$ be a conformal map. Then*

$$\begin{aligned} \dim_{\mathcal{H}} \left(\tilde{\Theta}^s(V) \setminus \phi^{-1}(\partial D) \right) &= \dim_{\mathcal{H}} \left(\tilde{\Theta}^{s \geq}(V) \setminus \phi^{-1}(\partial D) \right) = \tilde{\xi}(s), & 0 \leq s \leq s_+ \\ \dim_{\mathcal{H}} \left(\tilde{\Theta}^s(V) \setminus \phi^{-1}(\partial D) \right) &= \dim_{\mathcal{H}} \left(\tilde{\Theta}^{s \leq}(V) \setminus \phi^{-1}(\partial D) \right) = \tilde{\xi}(s), & s_- \leq s \leq 0 \\ \dim_{\mathcal{H}} \left(\Theta^s(V) \setminus \partial D \right) &= \dim_{\mathcal{H}} \left(\Theta^{s \geq}(V) \setminus \partial D \right) = \xi(s), & \frac{\kappa}{4} \leq s \leq s_+ \\ \dim_{\mathcal{H}} \left(\Theta^s(V) \setminus \partial D \right) &= \dim_{\mathcal{H}} \left(\Theta^{s \leq}(V) \setminus \partial D \right) = \xi(s), & s_- \leq s \leq \frac{\kappa}{4} \end{aligned}$$

That is, the conclusion of Theorem 1.1 holds a.s. away from the domain boundary at all times simultaneously for an $\text{SLE}_\kappa(\underline{\rho})$ with a general $\kappa > 0$ and vector of weights $\underline{\rho}$ up until the process either hits an interior force point or the continuation threshold.

Proof. This follows from Theorem 1.1 combined with Proposition 2.16 below. Note that the functions $\tilde{\xi}(s)$ and $\xi(s)$ are unaffected if we replace κ by $16/\kappa$, as one would expect from SLE duality [Zha08a, Zha10, Dub09a, MS16a, MS13]. \square

Remark 1.4. We believe that the techniques developed in this paper could also be employed to describe the multifractal behavior of the $\text{SLE}_\kappa(\underline{\rho})$ processes even near their intersection points with the domain boundary and near their tip, though we will not carry this out here.

Roughly speaking, the *harmonic measure spectrum* of a hull $A \subset \mathbf{H}$ gives, for each $\alpha \in (1/2, \infty)$, the Hausdorff dimension of the set $\Theta_{\text{hm}}^\alpha(A)$ of points $x \in \partial A$ for which the harmonic measure from ∞ of $B_\epsilon(x)$ in $\mathbf{H} \setminus A$ decays like ϵ^α as $\epsilon \rightarrow 0$ (or in the pre-image $\tilde{\Theta}_{\text{hm}}^\alpha(A)$ of $\Theta_{\text{hm}}^\alpha(A)$ under a conformal map $\mathbf{D} \rightarrow \mathbf{H} \setminus A$).

In [JVL12, Section 2.3], the authors give a rigorous treatment of the harmonic measure spectrum at the tip of a curve. A nearly identical construction works for the harmonic measure spectrum of a whole hull in \mathbf{H} . Similar constructions also work for hulls in \mathbf{D} or \mathbf{C} . In particular, one has (see [JVL12, Lemma 2.3])

$$\Theta^s(A) = \Theta_{\text{hm}}^{\frac{1}{1-s}}(\mathbf{H} \setminus A) \quad \forall s \in (-1, 1). \quad (1.7)$$

Remark 1.5. In light of the relationship between SLE_6 and Brownian motion [LSW01a], we see that Corollary 1.3 with $\kappa = 6$ yields the harmonic measure spectrum for the Brownian frontier computed in [Law96, LSW01a, LSW01b, LSW01c, LSW02].

Remark 1.6. In [Dup00] (see in particular [Dup00, Equation 6]), Duplantier predicts that the harmonic measure spectrum for the bulk of the SLE_κ curve is given by

$$f(\alpha) = \alpha + \frac{25-c}{24} \left(1 - \frac{1}{2} \left(2\alpha - 1 + \frac{1}{2\alpha - 1} \right) \right), \quad (1.8)$$

where

$$c = \frac{(6-\kappa)(6-16/\kappa)}{4}$$

is the central charge. The exponent (1.4) is related to the exponent (1.8) by

$$\xi(s) = f\left(\frac{1}{1-s}\right).$$

This is what we would expect in light of (1.7).

The dimension $\xi(s)$ attains a unique maximum value of $1 + \kappa/8$ on $[-1, 1]$ at $s = \kappa/4$. This maximum value coincides with the Hausdorff dimension of the SLE_κ curve [Bef08], which suggests that near a “typical point” of η , the modulus of the derivative of a conformal map from D_η to \mathbf{D} grows like $\text{dist}(z, \eta)^{\frac{\kappa}{4-\kappa}}$. Hence Theorem 1.1 gives an alternative proof of the following.

Corollary 1.7. *Let $\kappa \leq 4$. The Hausdorff dimension of an SLE_κ curve η is a.s. equal to $1 + \kappa/8$.*

We remark that we believe that the methods that we use to establish the lower bound in Theorem 1.1 could be employed to give an independent derivation of the lower bound of the dimension of SLE_κ for all $\kappa > 0$, however we will not carry this out here.

1.3 Integral means spectrum

The *integral means spectrum* of a simply connected domain $D \subset \mathbf{D}$ is the function $\text{IMS}_D : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$\text{IMS}_D(a) := \limsup_{\epsilon \rightarrow 0} \frac{\log \int_{\partial B_{1-\epsilon}(0)} |\phi'(z)|^a dz}{-\log \epsilon}, \quad (1.9)$$

where $\phi : \mathbf{D} \rightarrow D$ is a conformal map. (There is a three parameter family of such conformal maps, but $\text{IMS}_D(a)$ does not depend on the specific choice of ϕ .) The integral means spectrum is of substantial interest in complex analysis, primarily in the form of the *universal integral means spectrum*, which is defined by

$$\text{IMS}^U(a) := \sup_D \text{IMS}_D(a)$$

where the supremum is over all simply connected domains $D \subset \mathbf{C}$. It has been conjectured by Kraetzer [Kra96] that $\text{IMS}^U(a) = t^2/4$ for $|t| \leq 2$ and $\text{IMS}^U(a) = |t| - 1$ for $|t| \geq 2$. This conjecture has several important consequences in complex analysis. See, e.g., [Pom97, BS05, HS08, Pom92] for more details. The integral means spectrum is often very difficult to compute in practice for deterministic domains. However, domains bounded by random fractals (e.g. the complement of an SLE_κ curve) are sometimes more tractable. For example, in [BS09] Beliaev and Smirnov give an explicit calculation of the average integral means spectrum of the complement of a whole plane SLE_κ curve (which is defined as in (1.9) but with $|\phi'(z)|^a$ replaced by $\mathbf{E}(|\phi'(z)|^a)$).

In this paper we shall be interested in a slight refinement of the definition of the integral means spectrum for the complement of a curve which negates possible pathologies arising from unusual behavior at its endpoints or when it intersects itself or the boundary of the domain. Namely, let $D \subset \mathbf{C}$ be a bounded simply connected domain with smooth boundary and let $\eta : [0, T] \rightarrow \overline{D}$ be a non-self-crossing curve (we allow $T = \infty$). Let V be a connected component of $D \setminus \eta$. Let x_V be the first (equivalently last) point of ∂V hit by η and let $\phi : \mathbf{D} \rightarrow V$ be a conformal map.

For $\zeta > 0$, let

$$I^\zeta(\phi) := \phi^{-1}(\partial V \setminus (B_\zeta(\eta(T)) \cup B_\zeta(x_V) \cup B_\zeta(\partial D))). \quad (1.10)$$

Let $A_\epsilon^\zeta(\phi)$ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in I^\zeta(\phi)$. The *bulk integral means spectrum* of V is the function $\text{IMS}_V : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$\text{IMS}_V^{\text{bulk}}(a) := \sup_{\zeta > 0} \limsup_{\epsilon \rightarrow 0} \frac{\log \int_{A_\epsilon^\zeta(\phi)} |\phi'(z)|^a dz}{-\log \epsilon}. \quad (1.11)$$

One can check that the definition (1.11) does not depend on the choice of ϕ .

We extract the following from the proof of Theorem 1.1.

Corollary 1.8. *For $a \in \mathbf{R}$ with $a < \frac{(4+\kappa)^2}{8\kappa}$, let*

$$s_*(a) := -1 + \frac{4 + \kappa}{\sqrt{(4 + \kappa)^2 - 8a\kappa}}. \quad (1.12)$$

Also let s_- and s_+ be as in (1.5) and (1.6) and let a_- (resp. a_+) be the value of a for which $s_(a) = s_-$ (resp. $s_*(a) = s_+$). Set*

$$\text{IMS}^*(a) := \begin{cases} -1 + s_- a, & a < a_- \\ -a + \frac{(4 + \kappa)(4 + \kappa - \sqrt{(4 + \kappa)^2 - 8a\kappa})}{4\kappa}, & a \in [a_-, a_+] \\ -1 + s_+ a, & a > a_+. \end{cases} \quad (1.13)$$

Suppose we are in the setting of Corollary 1.3. Almost surely, the following is true. Let $a \in \mathbf{R}$ and let V be a complementary connected component of either $D \setminus \eta$ or of $D \setminus \eta^t$ for any $t > 0$ (before η hits an interior force point or the continuation threshold if it is an $\text{SLE}_\kappa(\rho)$ process). Then

$$\text{IMS}_V^{\text{bulk}}(a) = \text{IMS}^*(a). \quad (1.14)$$

The result of Corollary 1.8 is in agreement with the (rigorously proven) formula¹ for the average bulk integral means spectrum of whole-plane SLE in [BS09, Theorem 1] for $a \in [a_-, a_+]$, and with [BS09, Conjecture 1] for the a.s. bulk integral means spectrum for all values of $a \in \mathbf{R}$.

Remark 1.9. As conjectured in [BS09], the a.s. bulk integral means spectrum of Corollary 1.8 differs from the average integral means spectrum computed in [BS09] for values of $a \notin [a_-, a_+]$. We explain why this is the case. First, as noted in [BS09], we expect the average and a.s. bulk integral means spectra to differ because the function which gives the average bulk integral means spectrum does not satisfy Makarov's [Mak98] characterization of possible integral means spectra. At a more heuristic level, the average integral means spectrum for $a \notin [a_-, a_+]$ is distorted by the occurrence of the small (but still positive) probability event that a conformal map $\phi : \mathbf{D} \rightarrow V$ satisfies $|\phi'(z)| \approx (1 - |z|)^{-s}$ for some z close to $\partial \mathbf{D}$ and some $s \notin [s_-, s_+]$. However, this event a.s. does not occur in the limit (c.f. Theorem 1.1) so does not affect the a.s. bulk integral means spectrum.

¹The formula appearing in [BS09, Theorem 1] for the bulk integral means spectrum is actually equal to 5 plus the formula (1.13); the 5 in their formula is a misprint.

1.4 Outline

There is a systematic approach to computing Hausdorff dimensions of random fractal sets of the sort we consider here. One first gets a sharp estimate for the probability that a single point is contained in the set (the “one-point estimate”) and uses this to get an upper bound on the Hausdorff dimension. One then defines a subset of the set of interest (the “perfect points”) and obtains an estimate for the probability that any two given points are perfect (the “two-point estimate”). This enables one to define a Frostman measure on the set of perfect points and thereby obtain a lower bound on the Hausdorff dimension of the set of interest (see [MP10, Section 4] for more on Frostman measures and their connection to Hausdorff dimension). We will follow this outline here. See, e.g., [MW14, MWW14, JVL12, MSW14] for more examples of this technique.

We will now give a moderately detailed outline of the remainder of this paper. The reader should note that this section does not constitute a precise description of all of the proofs in our paper, but rather is only a heuristic guide. For the sake of brevity, many technical details have been omitted, especially in regards to proof of the two-point estimate.

In Section 2, we will give some background on the objects which appear in our proofs, including SLE, the GFF, and the various couplings between them. We will also establish some notations and prove some elementary lemmas which we will need in the sequel.

Next we will prove our one-point estimate. This is done in two stages. In Section 3, we will establish pointwise derivative estimates for the inverse centered Loewner maps (f_t^{-1}) for an SLE_κ . Roughly, our estimates will take the form

$$\mathbf{P}(|(f_t^{-1})'(z)| \approx \epsilon^{-s}, \text{ regularity conditions}) \approx \epsilon^{\alpha(s)}, \quad \forall s \in (-1, 1), \quad \forall z \in \mathbf{H} \text{ with } \text{Im } z = \epsilon. \quad (1.15)$$

Here $\alpha(s) = \frac{(4+\kappa)^2 s^2}{8\kappa(1+s)}$. The proof of these estimates is based on a family of non-negative martingales for the reverse Loewner flow (g_t) , analogous to the martingales for the forward SLE_κ flow in [SW05, Section 5]. The reverse Loewner flow is of interest because we have $g_t \stackrel{d}{=} f_t^{-1}$ for each fixed t (see, e.g., [RS05, Lemma 3.1]). For a given $z \in \mathbf{H}$ with $\text{Im } z = \epsilon$, one can find a martingale M_t^z with the property that $M_t \mathbf{1}_{\underline{E}(z)} \approx \epsilon^{-\alpha(s)}$, where $\underline{E}(z)$ denotes the event in the probability in (1.15) with g_t in place of f_t^{-1} . We then arrive at

$$\mathbf{P}(\underline{E}(z)) \approx \epsilon^{\alpha(s)} \mathbf{P}_*^z(\underline{E}(z)),$$

where \mathbf{P}_*^z denotes the measure obtained by re-weighting the law of the original SLE_κ process by M (which will be the law of a reverse chordal $\text{SLE}_\kappa(\rho)$ for an appropriate ρ). Hence we just need to show $\mathbf{P}_*^z(\underline{E}(z))$ is uniformly positive, independent of ϵ . This is done in two steps. First, to obtain $\mathbf{P}_*^z(|g_t'(z)| \approx \epsilon^{-s}) \rightarrow 1$ as $\epsilon \rightarrow 0$, we use a coupling of g_t with a GFF together with a coordinate change argument similar in spirit to the proof of [MS16d, Theorem 8.1]. To obtain that the auxiliary regularity conditions hold with uniformly positive probability under \mathbf{P}_*^z , we use a combination of stochastic calculus, forward/reverse (in the sense of Loewner flows) SLE symmetry, and GFF coupling arguments.

In Section 4 we use the estimate of Section 3 to establish pointwise derivative estimates for the “time infinity” conformal map Ψ_η associated with an SLE_κ process η from $-i$ to i in the unit disk \mathbf{D} , defined as follows. Let D_η be the right connected component of $\mathbf{D} \setminus \eta$, as in Theorem 1.1. Let $\Psi_\eta : D_\eta \rightarrow \mathbf{D}$ be the unique conformal map fixing $-i$, i , and 1 . Our estimates for Ψ_η take the form

$$\mathbf{P}(\text{dist}(z, \eta) \approx \epsilon^{1-s}, |\Psi_\eta'(z)| \approx \epsilon^s, \text{ regularity conditions}) \approx \epsilon^{\gamma(s)}, \quad \forall s \in (-1, 1), \quad \forall z \in \mathbf{D} \quad (1.16)$$

where $\gamma(s) = \alpha(s) - 2s + 1$ and $\alpha(s)$ as above. The idea of the proof of (1.16) is as follows. First we observe using the Koebe quarter theorem that for each $\epsilon > 0$ and each $t > 0$, the set of points $\underline{A}_\epsilon(t)$ in \mathbf{D} for which the analogue of the event of (1.15) with \mathbf{D} in place of \mathbf{H} occurs is (approximately) the image under f_t of the set $A_\epsilon(t)$ of points in \mathbf{D} for which the event of (1.16) holds with Ψ_η replaced by f_t and η replaced by $\eta([0, t])$. Hence the estimate (1.15) together with an elementary change of variables yields $\mathbf{E}(\text{Area } A_\epsilon(t)) \approx \epsilon^{\gamma(s)}$. We are then left to (a) transfer this area estimate from finite time to infinite time and (b) argue that the probability of the event (1.16) does not depend too strongly on z . Both tasks will be accomplished by means of various conditioning arguments which rely crucially on the regularity conditions involved in the estimate (1.15).

In Section 5, we will use the estimates (1.15) and (1.16) to prove upper bounds for the Hausdorff dimensions of the sets $\tilde{\Theta}^{s;*}(D_\eta)$ and $\Theta^{s;*}(D_\eta)$, where $*$ stands for \geq or \leq as well as an upper bound for the bulk integral means spectrum of D_η , as claimed in Corollary 1.8.

In Section 6 we prove our two-point estimate. The first step of the proof is a slight modification of the estimate (1.16). Namely, let $\bar{\eta}$ denote the time reversal of η , which has the law of a chordal SLE $_\kappa$ from i to $-i$ [Zha08b]. Let τ_β (resp. $\bar{\tau}_\beta$) be the first time η (resp. $\bar{\eta}$) hits the ball of radius $e^{-\beta}$ centered at the origin. Let $\eta^{\tau_\beta} = \eta([0, \tau_\beta])$, $\bar{\eta}^{\bar{\tau}_\beta} = \bar{\eta}([0, \bar{\tau}_\beta])$, and let ϕ_β be the conformal map from $\mathbf{D} \setminus (\eta^{\tau_\beta} \cup \bar{\eta}^{\bar{\tau}_\beta})$ to \mathbf{D} which fixes $-i$, i , and 1. Then we will use the one-point estimate (1.16) to show

$$\mathbf{P}(|\phi'_\beta(z)| \approx e^{-\beta q}, \text{ regularity conditions}) \approx e^{-\beta \gamma^*(q)}, \quad \forall q \in (-1/2, \infty). \quad (1.17)$$

Here $q = s/(1-s)$ and $\gamma^*(q) = \gamma(s)/(1-s) = (q+1)\gamma(q)$, with γ as in (1.16).

The estimate (1.17) allows us to break the event that $|\Psi'_\eta(0)| \approx e^{-n\beta}$ down into several stages and estimate each individually. Indeed, if we apply a conformal map from $\mathbf{D} \setminus (\eta^{\tau_\beta} \cup \bar{\eta}^{\bar{\tau}_\beta})$ to \mathbf{D} which fixes 0, then the rest of the curve will be mapped to another curve whose law is the same as that of η (modulo perturbations of its endpoints, which can be dealt with by growing out a little bit more of the curve). In this manner we can construct two approximately independent events $E_{0,1}$ and $E_{0,2}$ whose intersection is contained in the event $\{|\Psi'_\eta(0)| \approx e^{-2\beta q}\}$. By iterating this procedure we construct a sequence of approximately independent events $E_{0,j}$ such that $|\Psi'_\eta(0)| \approx e^{-n\beta q}$ on $E_n(0) := \bigcap_{j=1}^n E_{0,j}$ and $\mathbf{P}(E_{z,j}) \approx e^{-\beta \gamma^*(q)}$.² We can similarly construct events $E_{z,j}$ and $E_n(z)$ for any $z \in \mathbf{D}$ by first mapping z to 0.

For the lower bound on $\dim_{\mathcal{H}} \Theta^s(D_\eta)$, the perfect points will be, roughly speaking, the set of $z \in \mathbf{D}$ for which $E_n(z)$ occurs for every $n \in \mathbf{N}$. In order to obtain a lower bound on the Hausdorff dimension of the set of perfect points, we need to estimate the probability that $E_n(z)$ and $E_n(w)$ both occur for $z, w \in \mathbf{D}$, depending on $|z-w|$. To this end, suppose $|z-w| \approx e^{-\beta k}$. We condition on the event $E_k(z)$, corresponding to what happens before we get near z and w . After we map out the part of the curve which is grown before the k th stage, z and w will be at constant order distance from each other. See Figure 1.2.

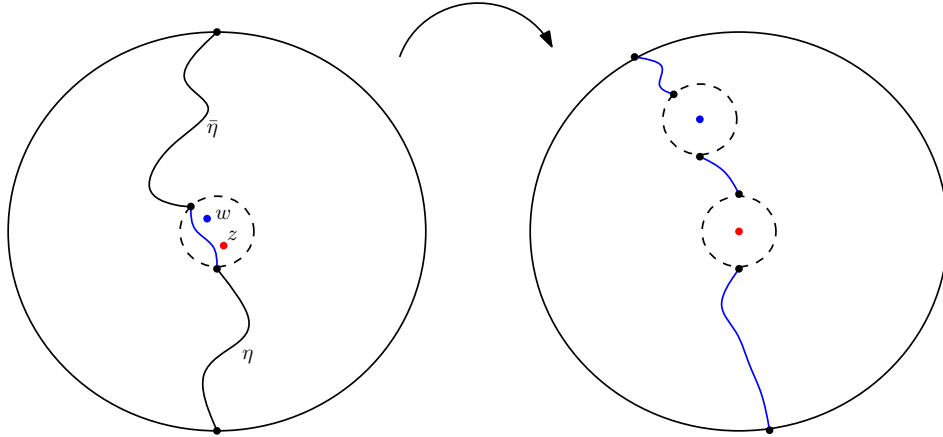


Figure 1.2: If $|z-w| \approx e^{-\beta k}$, then after applying a conformal map which takes the complement of the parts of η and $\bar{\eta}$ involved in the event $E_0^k(z)$ to \mathbf{D} and takes z to 0, the images of z and w will be at constant order distance from each other. Note, however, that in this setting the derivatives of the stage $k+1$ -map near z and w are not approximately independent, since they each depend on the whole curve in the picture on the right.

We would like to say that the behaviors of the curve near z and near w are approximately conditionally independent given $E_k(z)$. However, the derivatives of the maps we are interested in depend on the whole curve. Hence we need to localize our events. This is accomplished using a different coupling with a GFF,

²Actually, we will need to increase β by a little bit at each stage for technical reasons, but the basic idea of the argument is the same if we consider a fixed but large β .

namely the forward SLE/GFF coupling, or “imaginary geometry” coupling studied in [Dub09b, She16, She05, MS16a, MS16b, MS16c, MS13].

At each stage in the construction of the events $E_n(z)$, we can add auxiliary curves, which are all flow lines (in the sense of [MS16a]; c.f. Section 2.5) of the same GFF. These auxiliary curves will form pockets surrounding z with the property that the parts of η inside different pockets are independent once we condition on the pockets, and the derivative of Ψ_η at a point inside a pocket can be estimated by the derivative of a map which depends only on the behavior of η inside this pocket. We then define the event $E_{z,j}$ so that it depends only on the behavior of the curve inside the j th pocket. See Figure 1.3 for an illustration.

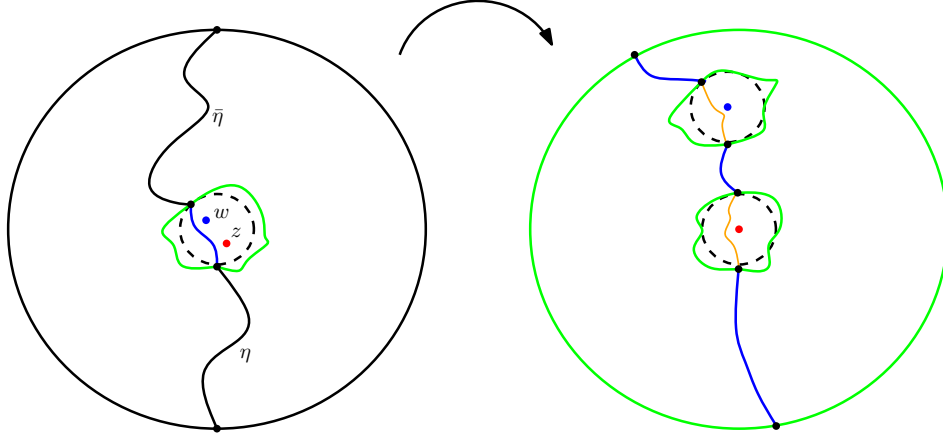


Figure 1.3: A modified version of Figure 1.2 where we add auxiliary curves (shown in green) at each stage to form a pocket. Here we define the events at each stage in terms of only the part of the curve inside the previous pocket. This gives us the needed local independence of the events $E_{z,j}$ and $E_{w,j}$.

The independence of the parts of η inside different pockets will eventually enable us to establish the two-point estimate needed for the proof of the lower bounds in Theorem 1.1.

In Section 7, we use our two-point estimate to prove lower bounds for the Hausdorff dimensions of the sets $\tilde{\Theta}^s(D_\eta)$ and $\Theta^s(D_\eta)$ as well as for the bulk integral means spectrum of D_η .

Appendix A contains the proof of an estimate which is needed in Section 3. Appendices B and C contain several technical lemmas which are needed primarily in Section 6.

2 Preliminaries

In this section we will establish some notations, give some background on the objects involved in the paper, and prove some elementary lemmas. We recommend that the reader familiarize themselves with Section 2.1 and Section 2.2 before reading the remainder of the paper, as the notations and results of these subsections will be used frequently in the sequel. Sections 2.3, 2.4, and 2.5 contain background on results on SLE, Gaussian free fields, and the couplings between them. Readers who are already familiar with these topics may wish to skim these subsections to acquaint themselves with the notations, and refer back to them as needed. Sections 2.6 and 2.7 contain some elementary lemmas about the sets whose Hausdorff dimensions we will compute. The results of these sections are not used extensively in the sequel, but are needed in Sections 5 and 7.

2.1 Basic notations

Given two variables a and b , we say $b = o_a(1)$ if $b \rightarrow 0$ as $a \rightarrow 0$ (or as $a \rightarrow \infty$, depending on the context) and we say $b = O_a(1)$ if b is bounded above by an a -independent constant for sufficiently small (or sufficiently small, depending on context) values of a . We usually allow $o_a(1)$ and $O_a(1)$ terms to depend on certain parameters other than a , but not on others. We will describe this dependence as needed.

We say that $a \preceq b$ (resp. $a \succeq b$) if there is a constant c which does not depend on the main parameters of interest such that $a \leq cb$ (resp. $a \geq cb$). We say $a \asymp b$ if $a \preceq b$ and $a \succeq b$. As in the case of $o_a(1)$ and $O_a(1)$ above, we usually allow the implicit constants in \preceq, \succeq , and \asymp to depend on certain parameters, but not on others, and we describe this dependence as needed.

For a point $z \in \mathbf{C}$ and $r > 0$, we write $B_r(z)$ for the ball of radius r centered at z . More generally, for a set $A \subset \mathbf{C}$, we write $B_r(A) = \bigcup_{z \in A} B_r(z)$.

For a curve $\eta : [0, T] \rightarrow \mathbf{C}$, we will often use the abbreviation

$$\eta^t = \eta([0, t]). \quad (2.1)$$

Furthermore, when there is no risk of ambiguity we will simply write η for the entire image of η .

For a domain D and $z \in D$, we write $\text{hm}^z(\cdot; D)$ for the harmonic measure from z in D . That is, for $A \subset \partial D$, $\text{hm}^z(A; D)$ is the probability that a Brownian motion started from z exits D in A .

If $D' = D \setminus \eta$ for some non-self-crossing curve $\eta \in \overline{D}$ and z is a point on η which is visited only once, we will write z^- (resp. z^+) for the prime end of D corresponding to the left (resp. right) side of z . When we use this notation, our curve η will have an obvious orientation and “left” and “right” are as viewed by someone walking along η in the forward direction.

We will also use the following notation.

Notation 2.1. Given a Jordan domain D and $x, y \in \partial D$, we write $[x, y]_{\partial D}$ for the closed counterclockwise arc from x to y in ∂D . We similarly define the open arc $(x, y)_{\partial D}$ and the half-open arcs $(x, y]_{\partial D}$ and $[x, y)_{\partial D}$.

2.2 Reverse continuity conditions

2.2.1 In the upper half plane

Here we introduce a regularity condition which will arise frequently in the remainder of the paper.

Definition 2.2. We denote by \mathcal{M} the set of increasing functions $\mu : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{\delta \rightarrow 0} \mu(\delta) = 0$.

Definition 2.3. Let f be a map from a subdomain D of \mathbf{H} into \mathbf{H} . For $\mu \in \mathcal{M}$, let $G(f, \mu)$ be the event that the following occurs. For any $\delta > 0$ and any $x, y \in \mathbf{R} \cap \partial D$ with $|x|, |y| \leq \delta^{-1}$ and $|x - y| \geq \delta$, we have $|f(x)|, |f(y)| \leq \mu(\delta)^{-1}$ and $|f(x) - f(y)| \geq \mu(\delta)$.

The statement that $\mathcal{G}(f, \mu)$ holds is the same as the statement that f^{-1} has a certain μ -dependent modulus of continuity on $f(\mathbf{R} \cup \infty)$, with $\mathbf{R} \cup \infty$ given the one-point compactification topology.

We note that

$$G(f, \mu_1) \cap G(g, \mu_2) \Rightarrow G(g \circ f, \mu_2 \circ \mu_1). \quad (2.2)$$

We are interested in the condition $G(f, \mu)$ (and the analogous conditions in the next subsection) for two reasons. The first is that these conditions imply bounds on the distance from certain subsets of ∂D to certain subsets of \mathbf{R} (or $\partial \mathbf{D}$ in the setting of the next subsection) and on its diameter (see Lemmas 2.4 and 2.8 below).

The second reason for our interest in the condition of Definition 2.3 is as follows. We will often want to study conformal maps which are normalized by specifying the images of certain marked boundary points. When composing various maps, our marked points might be mapped to somewhere other than where we want them to go. So, we will frequently need to apply a conformal automorphism (of \mathbf{D} or \mathbf{H}) at the end of our arguments to move the marked points to their desired positions. The condition $G(\cdot, \mu)$ ensures that the images of the marked points are not too close together, and so allows us to control the derivative of this conformal automorphism.

Lemma 2.4. Let η be a simple curve started from 0 in \mathbf{H} parametrized by capacity which does not hit \mathbf{R} . Write $\eta^t := \eta([0, t])$. Let $f_t : \mathbf{H} \setminus \eta^t \rightarrow \mathbf{H}$ be the centered Loewner maps for η , i.e. f_t is the time t Loewner map for η , minus a real number chosen so that it maps 0 to 0. Fix $T \in (0, \infty)$ and suppose that for some $\mu \in \mathcal{M}$, we have

$$f_T(-\delta) - f_T(0^-) \leq -\mu(\delta) \leq \mu(\delta) \leq f_T(\delta) - f_T(0^+), \quad \forall \delta > 0. \quad (2.3)$$

Then there is a $\mu' \in \mathcal{M}$ and a $d > 0$ depending only on μ and T such that

$$\text{diam } \eta^T \leq d \quad \text{and} \quad \forall \delta > 0, \forall z \in \eta^T \text{ with } |\text{Re } z| \geq \delta, \text{ we have } \text{Im } z \geq \mu'(\delta). \quad (2.4)$$

Conversely, if (2.4) holds for some $d > 0$ and some $\mu' \in \mathcal{M}$, we can find $\mu \in \mathcal{M}$ depending only on d and μ' such that $G(f_T, \mu)$ holds.

Note that it is clear that $G(f_T, \mu)$ implies (2.3), so Lemma 2.4 implies in particular that (2.4) holds for some d and μ' depending only on μ whenever $G(f_T, \mu)$ occurs.

Proof of Lemma 2.4. Let $\text{hm}_T^\infty = \text{hm}^\infty(\cdot; \mathbf{H} \setminus \eta^T)$ denote harmonic measure from ∞ in $\mathbf{H} \setminus \eta^T$, so for a set $I \subset \partial(\mathbf{H} \setminus \eta^T)$ (viewed as a collection of prime ends),

$$\text{hm}_T^\infty I := \lim_{y \rightarrow \infty} y \mathbf{P}^{iy}(B_\tau \in I)$$

for B a Brownian motion and τ its exit time from $\mathbf{H} \setminus \eta^T$. It follows from conformal invariance of Brownian motion that for any $I \subset \partial(\mathbf{H} \setminus \eta^T)$,

$$\text{hm}_T^\infty(I) = \frac{1}{\pi} \text{length } f_T(I), \quad (2.5)$$

where by length we mean Lebesgue measure.

Now, assume (2.3) holds. For any $r > 0$ and $x \in \mathbf{R}$, the harmonic measure from ∞ in \mathbf{H} of the line segment $[x, x + ir]$ from x to $x + ir$ is a constant depending only on r . For $\delta > 0$, we can find $r = r(\delta) > 0$ such that this constant is $< \pi\mu(\delta)$. If η^T contains a point $x + iy$ with $x \geq \delta$ and $y \leq r$, then $\text{hm}_T^\infty([0, \delta]) \leq \text{hm}_T^\infty([x, x + ir]) < \pi\mu(\delta)$. This contradicts our hypothesis on (2.3) and the relation (2.5). A similar statement holds if we instead consider $x \leq -\delta$. Hence each point of η^T with real part $\geq \delta$ in absolute value has imaginary part $\geq r$. This proves the second part of (2.4) with $\mu'(\delta) = r$.

For the first part of (2.4), fix $\delta > 0$. Denote by S_δ the set of points in $z \in \mathbf{H}$ with $|\text{Re } z| \geq \delta$. By the second part of (2.4), we have

$$\text{hm}_T^\infty(\eta^T \cap S_\delta) \leq \frac{1}{\mu'(\delta)} \lim_{y \rightarrow \infty} y \mathbf{E}^{iy}(\text{Im } B_\tau \mathbf{1}_{(B_\tau \in \eta^T \cap S_\delta)}). \quad (2.6)$$

By [Law05, Proposition 3.38] we have

$$T = \text{hcap } \eta^T = \lim_{y \rightarrow \infty} y \mathbf{E}^{iy}(\text{Im } B_\tau) \quad (2.7)$$

so (2.6) is at most $T/\mu'(\delta)$. On the other hand, (2.7) and the Beurling estimate imply that $\sup_{z \in \eta^T} \text{Im } z$ is bounded above by a constant C_0 depending only on T . The harmonic measure from ∞ in \mathbf{H} of $[-\delta, \delta] \times [0, C_0]$ is at most a constant C_1 depending only on δ and T . Therefore

$$\text{hm}_T^\infty(\eta^T) \leq T/\mu'(\delta) + C_1.$$

By [Law05, equation 3.13], this implies $\text{diam } \eta^T$ is bounded above by a constant depending only on μ and T .

Conversely, suppose (2.4) holds. For $\delta > 0$, let U_δ be the set of points in $z \in \mathbf{H}$ with $|z| \leq d$ and either $|\text{Re } z| \leq \delta/2$ or $\text{Im } z \geq \mu'(\delta/2)$. Then $\eta^T \subset U_\delta$. The harmonic measure from ∞ of each sub-interval of $[\delta/2, \delta^{-1}] \cup [-\delta^{-1}, -\delta/2]$ in $\mathbf{H} \setminus U_\delta$ of length $\delta/2$ is at least some constant $\mu_0(\delta)$ depending only on δ and $\mu'(\delta/2)$. By (2.5), this implies that the length of the image of such an interval under f_T is at least a $\pi\mu_0(\delta)$. On the other hand, [Law05, Proposition 3.46] implies that we can find $\mu_1(\delta) > 0$ depending only on δ and d such that $|f_T(x)| \leq \mu_1(\delta)^{-1}$ for each $x \in [-\delta^{-1}, \delta^{-1}]$. This proves that $\mathcal{G}(f_T, \mu)$ holds with $\mu = (\pi\mu_0) \vee \mu_1$. \square

2.2.2 In the disk

The following is the analogue of Definition 2.3 for the unit disk \mathbf{D} .

Definition 2.5. Let $D \subset \mathbf{D}$ be a subdomain and let $I \subset \partial\mathbf{D} \cap \partial D$. Let $f : D \rightarrow \mathbf{D}$ be a conformal map. Let $\mu \in \mathcal{M}$ (Definition 2.2). We say that $\mathcal{G}_I(f, \mu)$ occurs if the following is true. For each $\delta > 0$ and each $x, y \in I$ with $|x - y| \geq \delta$, we have $|f(x) - f(y)| \geq \mu(\delta)$. We abbreviate

$$\mathcal{G}(f, \mu) = \mathcal{G}_{\partial\mathbf{D} \cap \partial D}(f, \mu).$$

We also make the following definition.

Definition 2.6. Let $A \subset \overline{\mathbf{D}}$ be a closed set and $I \subset \overline{\partial\mathbf{D} \setminus A}$. (Oftentimes we will take I to be a closed arc with endpoints in A , or a finite union of such arcs.) We say that $\mathcal{G}'_I(A, \mu)$ occurs if the following is true. For each $\delta > 0$, A lies at distance at least $\mu(\delta)$ from $I \setminus B_\delta(I \cap A)$. We write

$$\mathcal{G}'(A, \mu) = \mathcal{G}_{\overline{\partial\mathbf{D} \setminus A}}(A, \mu).$$

Remark 2.7. We will frequently find ourselves in the following situation. Suppose we are given a deterministic arc $I \subset \partial\mathbf{D}$, a random closed subset $A \subset \overline{\mathbf{D}}$ with $I \subset \overline{\partial\mathbf{D} \setminus A}$ a.s., and a deterministic $\epsilon > 0$. In this case we can find (using monotonicity) a deterministic $\mu \in \mathcal{M}$ for which $\mathbf{P}(\mathcal{G}_I(A, \mu)) \geq 1 - \epsilon$ where \mathbf{P} is typically the law of SLE.

The conditions of Definitions 2.5 and 2.6 will serve as the main “global regularity” conditions in our estimates starting from Section 4. The relationship between the conditions $\mathcal{G}(\cdot)$ and $\mathcal{G}'(\cdot)$ is contained in the following lemma.

Lemma 2.8. *Let $A \subset \overline{\mathbf{D}}$ be a closed set and $I = [x, y]_{\partial\mathbf{D}}$ be an arc contained in $\overline{\partial\mathbf{D} \setminus A}$. Let $m \in (x, y)_{\partial\mathbf{D}}$ and suppose that $|x - m|$ and $|y - m|$ are each at least $\Delta > 0$. Let D be the connected component of $\mathbf{D} \setminus A$ containing I on its boundary. Let $\Phi : D \rightarrow \mathbf{D}$ be the unique conformal map taking x to $-i$, y to i , and m to 1 .*

1. *For each $\mu \in \mathcal{M}$, there exists $\mu' \in \mathcal{M}$ depending only on μ and Δ such that if $\mathcal{G}_I(\Phi, \mu)$ occurs, then $\mathcal{G}'_I(A, \mu')$ occurs.*
2. *Conversely, suppose $I' \subset I$ and $\mathcal{G}'_{I'}(A, \mu)$ holds for some $\mu \in \mathcal{M}$. There is a $\mu' \in \mathcal{M}$ depending only on μ and Δ such that $\mathcal{G}_{I'}(\Phi, \mu')$ holds. In fact, the following superficially stronger statement is true. For each $\delta > 0$, Φ is Lipschitz continuous on $I' \setminus (B_\delta(x) \cup B_\delta(y))$ and Φ^{-1} is Lipschitz continuous on $\Phi(I' \setminus (B_\delta(x) \cup B_\delta(y)))$ with Lipschitz constants depending only on $\mu(\delta)$, δ , and Δ .*

Proof. The basic idea of the proof is similar to that of Lemma 2.4, but we consider harmonic measure from m rather than harmonic measure from ∞ .

Fix $\delta > 0$. Let x_δ and y_δ be the unique points of I lying at distance δ from x and y , respectively. Let \widehat{D} be the radial reflection of D across I , viewed as a subset of the Riemann sphere. Extend Φ to \widehat{D} by Schwarz reflection. Then Φ maps \widehat{D} into $\mathbf{C} \setminus [i, -i]_{\partial\mathbf{D}}$, and maps I to $[-i, i]_{\partial\mathbf{D}}$. Suppose $\delta > 0$. Let $\widehat{D}_\delta = \widehat{D} \setminus [y_\delta, y]_{\partial\mathbf{D}}$. Let $\tilde{y}_\delta := \Phi(y_\delta)$. Then \tilde{y}_δ is determined by the condition that the harmonic measure of $[y_\delta, i]_{\partial\mathbf{D}}$ from m in \widehat{D}_δ equals the harmonic measure of the side of $[\tilde{y}_\delta, i]_{\partial\mathbf{D}}$ closer to 0 from 1 in $(\mathbf{C} \cup \infty) \setminus [y'_\delta, -i]_{\partial\mathbf{D}}$.

If $\mathcal{G}_I(\Phi, \mu)$ occurs, then \tilde{y}_δ lies at distance at least $\mu(\delta)$ from i , which means that the harmonic measure of $[y_\delta, y]_{\partial\mathbf{D}}$ from 1 in \widehat{D}_δ is at least some constant $\epsilon > 0$ depending only on $\mu(\delta)$. By symmetry, the same holds for $[x, x_\delta]_{\partial\mathbf{D}}$.

By the Beurling estimate, we can find some $\zeta_0 > 0$ depending only on ϵ such that $\text{dist}(m, A) \geq \zeta_0$. We can find a $\zeta_1 > 0$ such that if $z \in [x_\delta, y_\delta]_{\partial\mathbf{D}}$ lies at distance at least ζ_0 from m , then the probability that a Brownian motion started from m hits $B_{\zeta_1}(z)$ before hitting $[i, -i]_{\partial\mathbf{D}}$ is at most ϵ . If $\text{dist}(z, A) < \zeta_1$ for such a z , then a Brownian motion started from 1 must hit $B_{\zeta_1}(z)$ before hitting either $[y_\delta, y]_{\partial\mathbf{D}}$ or $[x, x_\delta]_{\partial\mathbf{D}}$. Hence we must have $\text{dist}(z, A) \geq \zeta_1 \wedge \zeta_0$ for each $z \in [x_\delta, y_\delta]_{\partial\mathbf{D}}$. This proves assertion 1 with $\mu'(\delta) = \zeta_1 \wedge \zeta_0$.

Conversely, suppose $I' \subset I$ and $\mathcal{G}'_{I'}(A, \mu)$ holds for some $\mu \in \mathcal{M}$. For $\delta > 0$ let x'_δ be either x_δ (as defined just above) or the endpoint of I' closest to x , whichever is closest to x . Define y'_δ similarly. A Brownian motion started from any point of $[x'_\delta, y'_\delta]_{\partial\mathbf{D}}$ as a positive probability depending only on δ , $\mu(\delta)$, and Δ to stay within distance $\mu(\delta)$ of I until it hits $[y'_\delta, y]_{\partial\mathbf{D}}$ (resp. $[x, x'_\delta]_{\partial\mathbf{D}}$). By the Beurling estimate there is a $\mu_0(\delta)$ depending only on $\mu(\delta)$, δ , and Δ such that $\Phi([x'_\delta, y'_\delta]_{\partial\mathbf{D}})$ lies at distance at least $\mu_0(\delta)$ from $[i, -i]_{\partial\mathbf{D}}$.

It remains to establish the Lipschitz continuity statement. For this, we observe that for any $z \in [x'_\delta, y'_\delta]_{\partial\mathbf{D}}$, the Koebe quarter theorem implies

$$\frac{\text{dist}(\Phi(z), [i, -i]_{\partial\mathbf{D}})}{4 \text{dist}(z, A) \wedge \delta} \leq |\Phi'(z)| \leq \frac{4 \text{dist}(\Phi(z), [i, -i]_{\partial\mathbf{D}})}{\text{dist}(z, A) \wedge \delta}.$$

Hence

$$\frac{\mu_0(\delta)}{8} \leq |\Phi'(z)| \leq \frac{8}{\mu(\delta) \wedge \delta}.$$

So, $|\Phi'|$ is bounded above and below by positive constants on $[x'_\delta, y'_\delta]_{\partial\mathbf{D}}$ depending only on $\mu(\delta)$, δ , and Δ which establishes the desired Lipschitz continuity. \square

2.3 Schramm-Loewner evolution

Let $t \mapsto W_t$ be a continuous function on $[0, \infty)$. The *chordal Loewner equation* is the ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z. \quad (2.8)$$

A solution to (2.8) is a family of conformal maps $\{g_t : t \geq 0\}$ from subdomains of \mathbf{H} to \mathbf{H} , satisfying the hydrodynamic normalization $\lim_{z \rightarrow \infty} (g_t(z) - z) = 0$. The complements (K_t) of the domains of (g_t) in \mathbf{H} are an increasing family of closed subsets of \mathbf{H} called the *hulls* of the process. The *centered Loewner maps* corresponding to (g_t) are defined by

$$f_t := g_t - W_t.$$

A chordal *Schramm-Loewner evolution* with parameter $\kappa > 0$ (SLE_κ) is the random evolution obtained by solving (2.8) where the driving process W is $\sqrt{\kappa}$ times a Brownian motion. It can be shown [RS05] that this Loewner evolution is generated by a curve which we typically denote by η . Chordal SLE_κ on other domains is defined by conformal mapping. We refer the reader to [Law05] or [Wer04] for a more detailed introduction to SLE.

More generally, suppose we are given a vector of real weights $\bar{\rho} = (\rho^1, \dots, \rho^n)$ and a collection of points $z^1, \dots, z^n \in \mathbf{H}$. Chordal $\text{SLE}_\kappa(\bar{\rho})$ is the random evolution obtained by solving (2.8) with the driving function W part of the solution to the system of SDE's

$$dW_t = \sqrt{\kappa} dB_t + \sum_{i=1}^n \text{Re} \frac{\rho^i}{W_t - V_t^i} dt, \quad dV_t^i = \frac{2}{V_t^i - W_t} dt, \quad W_0 = y \quad V_0^i = z^i. \quad (2.9)$$

The points z^i are called the *force points*. See [LSW03, SW05, MS16a] for more on $\text{SLE}_\kappa(\bar{\rho})$.

We will also need to consider the *reverse Loewner equation*. This is the ODE

$$\partial_t g_t(z) = -\frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \quad (2.10)$$

whose solution is a family of conformal maps from \mathbf{H} to sub-domains of \mathbf{H} . Reverse SLE_κ is obtained by taking W_t to be $\sqrt{\kappa}$ times a Brownian motion. For each time t , the time t centered Loewner map of a reverse SLE_κ has the same law as the inverse of the time t centered Loewner map of a forward SLE_κ [RS05, Lemma 3.1].

We will also need to consider reverse $\text{SLE}_\kappa(\bar{\rho})$ with force points z^1, \dots, z^n , which is obtained by solving (2.10) with the driving function W part of the solution to the system of SDE's

$$dW_t = \sqrt{\kappa} dB_t + \sum_{i=1}^n \text{Re} \frac{\rho^i}{W_t - V_t^i} dt, \quad dV_t^i = -\frac{2}{V_t^i - W_t} dt, \quad W_0 = y \quad V_0^i = z^i.$$

For a general $\bar{\rho}$ we do not have as simple a relation between forward and reverse $\text{SLE}_\kappa(\bar{\rho})$ as we do for ordinary SLE_κ . However, there are various forward and reverse symmetries, some of which are discussed in [DMS14, She16].

Throughout most of the rest of this paper we will fix $\kappa \in (0, 4]$ and we will not always make dependence on κ explicit.

2.4 Gaussian free fields

For some of our results, we will make use of couplings of SLE_κ with Gaussian free fields. In this section we give some basic background about the latter object.

Let D be a domain in \mathbf{C} with harmonically non-trivial boundary (i.e. a Brownian motion started in D a.s. exits D in finite time). We denote by $H(D)$ the Hilbert space completion of the subspace of $C^\infty(\overline{D})$ consisting of those smooth, real-valued functions f such that

$$\int_D |\nabla f(z)|^2 dz < \infty, \quad \int_D f(z) dz = 0$$

with respect to the Dirichlet inner product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) dz. \quad (2.11)$$

A *free-boundary Gaussian free field* (GFF) on D is a random distribution (in the sense of Schwartz) on D given by the formal sum

$$h = \sum_{j=1}^{\infty} X_j f_j \quad (2.12)$$

where $\{f_j\}$ is an orthonormal basis for $H(D)$ and (X_j) is a sequence of i.i.d. standard Gaussian random variables. It is defined as a pointwise function, but for each $g \in H(D)$, the formal inner product

$$(h, g)_\nabla = \sum_{j=1}^{\infty} (f_j, g)_\nabla$$

converges almost surely. Moreover, (h, g) is a.s. defined for each fixed $g \in L^2(D)$ by the formula

$$(h, g) = (h, -\Delta^{-1}g)_\nabla \quad (2.13)$$

where Δ^{-1} denotes the inverse Laplacian with Neumann boundary conditions. More generally, this formula makes sense if g is any distribution whose inverse Laplacian is in $H(D)$.

Similarly, one can define a *zero-boundary GFF* on D by replacing $H(D)$ with $H_0(D)$, defined as the Hilbert space completion of the space of smooth compactly supported functions on D in the inner product (2.11). A zero boundary GFF is defined without the need to make a choice of additive constant. A Gaussian free field with a given choice of boundary data on ∂D is defined to be a zero boundary GFF plus the harmonic extension of the given boundary data to D .

If $V, V^\perp \subset H(D)$ are complementary orthogonal subspaces, then the formula (2.12) implies that h decomposes as the sum of its projections onto V and V^\perp . In particular, we can take V to be the closure $H_0(D)$ of $C_c^\infty(D)$ in the inner product (2.11) and V^\perp the set Harm_D of functions in $H(D)$ which are harmonic in D . This allows us to decompose a free boundary GFF as the sum of a zero boundary Gaussian free field and a random harmonic function \mathfrak{h} on D , the latter defined modulo additive constant. We call these distributions the *zero-boundary part* and *harmonic part* of h , respectively.

We refer to [She07] and the introductory sections of [SS13] and [MS16d] for more details on GFF's.

2.4.1 Reverse SLE/GFF coupling

The following relation between free boundary GFFs and reverse $\text{SLE}_\kappa(\rho)$ is established in [She16, Section 4.2]. Let (g_t) be the *centered* Loewner maps of a reverse $\text{SLE}_\kappa(\rho)$ with force points z^1, \dots, z^n as in Section 2.3. Let h be a free boundary GFF on \mathbf{H} , independent from (g_t) . For $t \geq 0$ let

$$h_t = h \circ g_t + \frac{2}{\sqrt{\kappa}} \log |g_t(\cdot)| + \frac{1}{2\sqrt{\kappa}} \sum_{i=1}^n \rho^i G(g_t(z^i), g_t(\cdot)),$$

where

$$G(x, y) := -\log |x - y| - \log |\bar{x} - y|$$

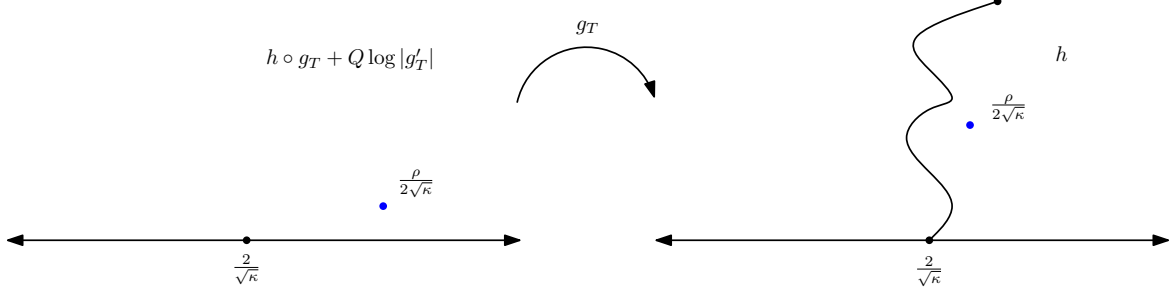


Figure 2.1: An illustration of the reverse SLE/GFF coupling in the case of a single force point (marked in blue). This is the case we will use in Section 3.

is the Green's function on \mathbf{H} with Neumann boundary conditions. Let

$$Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}. \quad (2.14)$$

Let τ be a stopping time for η which is a.s. less than the first time t that $f_t(z^i) = 0$ for some i . Then [She16, Theorem 4.5] implies that $h_\tau + Q \log |g'_\tau| \stackrel{d}{=} h_0$, modulo additive constant.

There is also an analogue of the above coupling for a zero boundary GFF paired with a forward $\text{SLE}_\kappa(\underline{\rho})$, which we discuss in Section 2.5.

2.4.2 Estimates for the harmonic part

In the course of proving our one-point estimate we will need some basic analytic lemmas about the harmonic part of a free boundary GFF which we will prove here.

Lemma 2.9. *Let \mathfrak{h} be the harmonic part of a free boundary GFF on \mathbf{D} , normalized so that $\mathfrak{h}(0) = 0$. Then for any $z, w \in \mathbf{D}$, $\mathfrak{h}(z)$ and $\mathfrak{h}(w)$ are jointly Gaussian with means zero and covariance*

$$\mathbf{E}(\mathfrak{h}(z)\mathfrak{h}(w)) = -2 \log |1 - z\bar{w}|.$$

Proof. For $n \geq 1$, let

$$\phi_n(z) = (2/n)^{1/2} \text{Re } z^n, \quad \psi_n(z) = (2/n)^{1/2} \text{Im } z^n. \quad (2.15)$$

Then $\{\phi_n, \psi_n : n \geq 1\}$ is an orthonormal basis for the set of harmonic functions on \mathbf{D} in the Dirichlet inner product. So, by definition of the free boundary GFF, we can write

$$\sum_{n=1}^{\infty} X_n \phi_n + \sum_{n=1}^{\infty} Y_n \psi_n, \quad (2.16)$$

where the X_n 's and Y_n 's are i.i.d. $N(0, 1)$. From this expression, it follows that $(\mathfrak{h}(z), \mathfrak{h}(w))$ is centered Gaussian for each $z, w \in \mathbf{D}$, and

$$\begin{aligned} \mathbf{E}(\mathfrak{h}(z)\mathfrak{h}(w)) &= \sum_{n=1}^{\infty} \phi_n(z)\phi_n(w) + \sum_{n=1}^{\infty} \psi_n(z)\psi_n(w) \\ &= 2 \sum_{n=1}^{\infty} \frac{(\text{Re } z^n)(\text{Re } w^n) + (\text{Im } z^n)(\text{Im } w^n)}{n} \\ &= \sum_{n=1}^{\infty} \frac{(z\bar{w})^n + (w\bar{z})^n}{n} \\ &= -\log(1 - z\bar{w}) - \log(1 - w\bar{z}) \\ &= -2 \log |1 - z\bar{w}|. \end{aligned}$$

□

Lemma 2.10. *Let h be a free boundary GFF on \mathbf{H} with additive constant chosen so that its harmonic part vanishes at a for some $a \in \mathbf{H}$. Let $A \subset \mathbf{H}$ be a deterministic hull lying at positive distance from a and let $g : \mathbf{H} \rightarrow \mathbf{H} \setminus A$ be the inverse centered hydrodynamic map. Let $\tilde{h} = h \circ g$ and let (\tilde{h}_ϵ) be the circle average process for \tilde{h} (see [DS11, Section 3.1] for more on the circle average process of a GFF). Fix $x \in \mathbf{R}$ and $\xi > 1/2$. For any $\delta \geq \epsilon > 0$, we have*

$$\mathbf{P}\left(|\tilde{h}_\epsilon(x + i\delta)| > (\log \epsilon^{-1})^\xi\right) = o_\epsilon(\epsilon^p) \quad \forall p > 0, \quad (2.17)$$

at a rate depending only on x , a , $\text{diam } A$, ξ , and δ , but uniform for x in compact subsets of \mathbf{R} , a in compact subsets of \mathbf{H} , and δ in compact subsets of $[\epsilon, \infty)$.

Proof. Write $h = h^0 + \mathfrak{h}$, for h^0 a zero boundary GFF and \mathfrak{h} an independent harmonic function. Let \mathfrak{h}_A be projection of h^0 onto the set of functions which are harmonic on $\mathbf{H} \setminus A$ and let $h_A^0 = h^0|_A - \mathfrak{h}_A$ be the zero-boundary part of $h^0|_A$. Then we can write

$$h|_{\mathbf{H} \setminus A} = h_A^0 + \mathfrak{h}_A + \mathfrak{h}|_{\mathbf{H} \setminus A}, \quad (2.18)$$

with the three summands independent. The function g increases imaginary parts, so it follows from Lemma 2.9 and a coordinate change to \mathbf{D} that $\mathfrak{h}(g(x + i\delta))$ is centered Gaussian with variance $\leq 2 \log \delta^{-1} + O_\epsilon(1)$.

By the Koebe distortion theorem, $|g'(x + i\delta)|$ is at least a constant depending only on y times $\delta|g'(x + iy)|$ for any $y > \delta$. By [Law05, Proposition 3.46] and the Koebe quarter theorem, for large enough y (depending only on $\text{diam } A$), $|g'(x + iy)|$ is bounded above by a constant depending only on $\text{diam } A$. By another application of the Koebe quarter theorem, we therefore have

$$\text{dist}(g(x + i\delta), A) \geq \delta^2. \quad (2.19)$$

It follows from [MS16a, Lemma 6.4] that $\mathfrak{h}_A(g(x + i\delta))$ is centered Gaussian with variance at most $2 \log \delta^{-1} + O_\epsilon(1)$.

By conformal invariance, $h_A^0 \circ g$ has the law of a zero boundary GFF on \mathbf{H} . By (2.19) and [DS11, Proposition 3.1], the circle average $(h_A^0 \circ g)_\epsilon(x + i\delta)$ is Gaussian with mean 0 and variance at most $2 \log \epsilon^{-1} + O_\epsilon(1)$. By (2.18),

$$\tilde{h}_\epsilon(x + i\delta) = (h_A^0 \circ g)_\epsilon(x + i\delta) + \mathfrak{h}_A(g(x + i\delta)) + \mathfrak{h}(g(x + i\delta))$$

is Gaussian with mean 0 and variance at most $6 \log \epsilon^{-1} + O_\epsilon(1)$. We obtain (2.17) from the Gaussian tail bound. \square

2.5 Imaginary geometry

The proof of the lower bounds in our main theorems will make heavy use of the so-called forward coupling of SLE_κ or $\text{SLE}_\kappa(\rho)$ with the GFF with Dirichlet boundary conditions. In this coupling, $\text{SLE}_\kappa(\rho)$ for $\kappa \in (0, 4)$ can be interpreted as the flow line of the formal vector field $e^{ih/\chi}$ where h is a GFF and

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}. \quad (2.20)$$

For $\kappa > 4$, $\text{SLE}_\kappa(\rho)$ can be interpreted as a “tree” or “light-cone” of $\text{SLE}_{16/\kappa}$ flow lines [MS16a]. The case $\kappa = 4$ is somewhat degenerate (though simpler to analyze) since $\chi \rightarrow 0$ as $\kappa \rightarrow 4$. $\text{SLE}_4(\rho)$ has the interpretation of being a level line (rather than a flow line or light cone) of the GFF. See [WW14] for a detailed study of this case.

The coupling of SLE_4 with the GFF was actually the first coupling in this family to be discovered [SS13] (see also [SS09] which gives the convergence of the contours of the discrete GFF to SLE_4). The existence of the forward coupling in the general setting is established in [Dub09b, SS13, She05, MS16a]; see [MS16a, Theorem 1.1] for a precise statement. The theory of how different flow lines and light cones of the same GFF interact is developed in [MS16a, MS16b, MS16c, MS13]; these works are also where the term “imaginary geometry” is coined. At this point in time, there are several places which contain short “crash courses” on imaginary geometry which are sufficient to understand its usage in this work. We refer the reader to one of [MS16b, Section 2.2], [MS13, Section 2.3], or [MW14, Section 2.2]; [MS16a, Section 1] and [MS13, Section 4] contain many of the main theorem statements in addition to more detailed overviews of the related literature.

2.6 Properties of the multifractal spectrum sets

In this subsection we will prove some elementary properties of the sets of Section 1.1, as well as a lemma which is relevant to the integral means spectrum. Our first lemma tells us that the sets of Section 1.1 are only non-empty in the case $s \in [-1, 1]$.

Lemma 2.11. *Let $D \subset \mathbf{C}$ be a simply connected domain and let $\phi : \mathbf{D} \rightarrow D$ be a conformal map. For each $x \in \partial\mathbf{D}$, there is a constant $C > 1$ depending only on ϕ and $\phi(x)$ but uniform for $\phi(x)$ in compact subsets of \overline{D} such that for each sufficiently small $\epsilon > 0$,*

$$C^{-1}\epsilon \leq |\phi'((1-\epsilon)x)| \leq C\epsilon^{-1}.$$

Proof. By the Cauchy estimate,

$$|\phi'((1-\epsilon)x)| \leq \epsilon^{-1} \sup_{z \in B_\epsilon((1-\epsilon)x)} |\phi(z)|$$

which gives the upper bound. For the lower bound, we apply the Koebe distortion theorem. \square

Next we prove some lemmas which give that the multifractal spectrum sets are invariant under reasonable modifications of the definitions.

Lemma 2.12. *Let $D \subset \mathbf{C}$ be a simply connected domain, $\phi : \mathbf{D} \rightarrow D$ a conformal map, and fix $x \in \partial\mathbf{D}$. Let $\gamma : [0, 1] \rightarrow \overline{\mathbf{D}}$ be a simple smooth curve such that $\gamma(0) = x$, $\gamma((0, 1]) \subset \mathbf{D}$, and $\gamma'(0)$ is not tangent to $\partial\mathbf{D}$ at x . Then*

$$\limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)x)|}{-\log \epsilon} = \limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'(\gamma(\epsilon))|}{-\log \epsilon}. \quad (2.21)$$

If one of the limsups is in fact a true limit, then the other is as well.

Proof. By Taylor's formula, we can write

$$\gamma(\epsilon) = x + \epsilon\gamma'(0) + O_\epsilon(\epsilon^2). \quad (2.22)$$

The function $w \mapsto \phi(\epsilon w + (1-\epsilon)x)/(\phi'((1-\epsilon)x)\epsilon)$ is a conformal map from \mathbf{D} into \mathbf{C} with unit derivative at the origin. By the Koebe distortion theorem applied to this map evaluated at $w = \epsilon^{-1}(\gamma(\epsilon) - (1-\epsilon)x)$ it follows that

$$1 - |\gamma'(0) + x + O_\epsilon(\epsilon)| \leq \frac{|\phi'(\gamma(\epsilon))|}{|\phi'((1-\epsilon)x)|} \leq \frac{1}{(1 - |\gamma'(0) + x + O_\epsilon(\epsilon)|)^3}. \quad (2.23)$$

Since $\gamma'(0)$ is not tangent to $\partial\mathbf{D}$ at x , there is some $c > 0$ such that $|c\gamma'(0) + x| < 1$. It follows that we can perform a linear re-parametrization of γ in such a way that $|\gamma'(0) + x| < 1$. Then (2.23) implies the statement of the lemma. \square

Lemma 2.13. *Let D and D' be two simply connected domains in \mathbf{C} , bounded by curves, which share a common boundary arc I . Let z be a prime end lying in the interior of I . Then for each $s \in \mathbf{R}$, we have $z \in \Theta^s(D)$ iff $z \in \Theta^s(D')$. The same holds with $\Theta^{s;\geq}(\cdot)$ or $\Theta^{s;\leq}(\cdot)$ in place of $\Theta^s(\cdot)$.*

Proof. By comparing D and D' to the connected component of $D \cap D'$ with I on its boundary, it suffices to consider the case where $D' \subset D$. Let $\phi : \mathbf{D} \rightarrow D$ and $\psi : \mathbf{D} \rightarrow D'$ be the corresponding conformal maps. We can factor $\phi = \psi \circ \xi$, where $\xi = \psi^{-1} \circ \phi$. Then

$$\phi'((1-\epsilon)\phi^{-1}(z)) = \psi'(\xi((1-\epsilon)\phi^{-1}(z)))\xi'((1-\epsilon)\phi^{-1}(z)) \quad (2.24)$$

By Schwarz reflection, ξ extends to be analytic in a neighborhood of $\phi^{-1}(z)$, so $|\xi'((1-\epsilon)\phi^{-1}(z))|$ is bounded above and below by positive constants for small ϵ . Let $\gamma(\epsilon) = \xi((1-\epsilon)\phi^{-1}(z))$. Note that γ is a simple curve in \mathbf{D} with $\gamma(0) = \psi^{-1}(z)$. We have

$$\gamma'(0) = -\xi'(\phi^{-1}(z))\phi^{-1}(z).$$

Since ξ maps a neighborhood of $\phi^{-1}(z)$ in $\partial\mathbf{D}$ into $\partial\mathbf{D}$, we have that $\xi'(\phi^{-1}(z))$ is a real multiple of $\frac{\xi(\phi^{-1}(z))}{\phi^{-1}(z)} = \frac{\psi^{-1}(z)}{\phi^{-1}(z)}$. Hence $\gamma'(0)$ is a real multiple of $\psi^{-1}(z)$. In particular γ is not tangent to $\partial\mathbf{D}$ at $\psi^{-1}(z)$ so the stated result follows from Lemma 2.12. \square

Lemma 2.14. *Let $D \subset \mathbf{C}$ be a simply connected domain. Let $\phi : \mathbf{D} \rightarrow D$ and $\psi : \mathbf{H} \rightarrow D$ be conformal maps. For each prime end $z \in \partial D$ with $\psi^{-1}(z) \neq \infty$, one has*

$$\limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)\phi^{-1}(z))|}{-\log \epsilon} = \limsup_{\epsilon \rightarrow 0} \frac{\log |\psi'(\psi^{-1}(z) + i\epsilon)|}{-\log \epsilon}. \quad (2.25)$$

If one of the limsups is in fact a true limit, then the other is as well.

Proof. We can write $\psi = \phi \circ \xi$ where $\xi = \phi^{-1} \circ \psi : \mathbf{H} \rightarrow \mathbf{D}$ is a conformal map. Then

$$\psi'(\psi^{-1}(z) + i\epsilon) = \phi'(\xi(\psi^{-1}(z) + i\epsilon))\xi'(\psi^{-1}(z) + i\epsilon).$$

The map ξ' extends smoothly to $\partial \mathbf{H}$, so $|\xi'(\psi^{-1}(z) + i\epsilon)|$ is bounded above and below by positive constants for small ϵ . Let $\gamma(\epsilon) = \xi(\psi^{-1}(z) + i\epsilon)$. Then $\gamma'(0) = i(\phi^{-1})'(z)/(\psi^{-1})'(z)$, which is not tangent to $\partial \mathbf{D}$ at $\phi^{-1}(z)$. Therefore the desired result follows from Lemma 2.12. \square

Lemma 2.14 implies in particular that if $\psi : \mathbf{H} \rightarrow D$ is a conformal map, then $\dim_{\mathcal{H}} \Theta^s(D)$ and $\dim_{\mathcal{H}} \tilde{\Theta}^s(D)$ are unaffected if we replace their definitions from Section 1.1 with

$$\tilde{\Theta}^s(D) = \left\{ x \in \mathbf{R} : \lim_{\epsilon \rightarrow 0} \frac{\log |\psi'(x + i\epsilon)|}{-\log \epsilon} = s \right\} \quad \text{and} \quad \Theta^s(D) = \psi(\tilde{\Theta}^s(D)) \quad (2.26)$$

The analogous statement also holds for the sets $\Theta^{s;\geq}(D)$, $\tilde{\Theta}^{s;\geq}(D)$, $\Theta^{s;\leq}(D)$, and $\tilde{\Theta}^{s;\leq}(D)$.

What follows is the analogue of Lemma 2.13 for the integral means spectrum.

Lemma 2.15. *Let D and D' be two bounded Jordan domains in \mathbf{C} and suppose there exists a connected boundary arc I shared by D and D' . Let $\phi : \mathbf{D} \rightarrow D$ and $\psi : \mathbf{D} \rightarrow D'$ be conformal maps. Let J' be a closed subset of the interior of I and let J be a closed subset of the interior of J' . For $\epsilon > 0$, let A_ϵ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in \phi^{-1}(J)$ and let A'_ϵ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in \psi^{-1}(J')$. Then we have*

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \int_{A_\epsilon} |\phi'(z)|^a dz}{-\log \epsilon} \leq \limsup_{\epsilon \rightarrow 0} \frac{\log \int_{A'_\epsilon} |\psi'(z)|^a dz}{-\log \epsilon}. \quad (2.27)$$

Proof. Let ξ be the conformal map from a subdomain of \mathbf{D} to a subdomain of $D' \cap D$ which equals $\psi^{-1} \circ \phi$ wherever the latter is defined. By Schwarz reflection ξ extends to a conformal map from a neighborhood of $\phi^{-1}(J')$ to a neighborhood of $\psi^{-1}(J')$. In particular $|\xi'| \asymp 1$ on a neighborhood of $\phi^{-1}(J')$, with implicit constants independent of ϵ . By a change of variables, for sufficiently small $\epsilon > 0$ we have

$$\int_{A_\epsilon} |\phi'(z)|^a dz \asymp \int_{A_\epsilon} |\psi'(\xi(z))|^a dz \asymp \int_{\xi(A_\epsilon)} |\psi'(w)| dw. \quad (2.28)$$

Let p_ϵ be the radial projection from \mathbf{D} onto $\partial B_{1-\epsilon}(0)$. By the above application of Schwarz reflection (and the fact that J is contained in the interior of J'), for sufficiently small $\epsilon > 0$, we have that p_ϵ restricts to a diffeomorphism from $\xi(A_\epsilon)$ to a subset \tilde{A}'_ϵ of A'_ϵ . Furthermore, since $|\xi'| \asymp 1$ on a neighborhood of $\psi^{-1}(J')$, we have $|p'_\epsilon| \asymp 1$ on $\xi(A_\epsilon)$ for sufficiently small ϵ , and by the Koebe distortion theorem we have $|\psi'(p_\epsilon(w))| \asymp |\psi'(w)|$ for $w \in \xi(A_\epsilon)$ and sufficiently small ϵ . Therefore, a second change of variables yields

$$\int_{\xi(A_\epsilon)} |\psi'(w)| dw \asymp \int_{\tilde{A}'_\epsilon} |\psi'(z)| dz \leq \int_{A'_\epsilon} |\psi'(z)| dz. \quad (2.29)$$

We obtain (2.27) by combining (2.28) and (2.29). \square

2.7 Zero-one laws

In this section we will prove that the multifractal spectrum and integral means spectrum of an $\text{SLE}_\kappa(\underline{\rho})$ curve are a.s. deterministic and do not depend on $\underline{\rho}$ or on which complementary component of the curve we consider.

Proposition 2.16. *Let $D \subset \mathbf{C}$ be a smoothly bounded domain. Let $\kappa > 0$ and let $\underline{\rho}$ be a vector of real weights. Let η be a chordal $\text{SLE}_\kappa(\underline{\rho})$ process in D , with any choice of initial and target points and force points located anywhere in \overline{D} , run up until the first time it either hits an interior force point or hits the continuation threshold after which it is no longer defined (c.f. [MS16a, Section 2.1]). Fix $s \in (-1, 1)$. Almost surely, the following is true. Let V be a connected component of $D \setminus \eta$ or a connected component of $D \setminus \eta([0, t])$ for any $t > 0$ and let $\phi : \mathbf{D} \rightarrow V$ be a conformal map. The Hausdorff dimension of each of the sets $\Theta^s(V) \setminus \partial D$, $\tilde{\Theta}^s(V) \setminus \phi^{-1}(\partial D)$, $\Theta^{s \geq}(V) \setminus \partial D$, $\tilde{\Theta}^{s \geq}(V) \setminus \phi^{-1}(\partial D)$, $\Theta^{s \leq}(V) \setminus \partial D$, and $\tilde{\Theta}^{s \leq}(V) \setminus \phi^{-1}(\partial D)$ is equal to a deterministic constant which depends only on κ and s . Furthermore, the a.s. Hausdorff dimensions of the corresponding sets for κ and $16/\kappa$ are equal.*

Proof. We begin with some reductions. By Lemma 2.14 we can consider SLE_κ from 0 to ∞ in \mathbf{H} instead of SLE_κ between two arbitrary points in D , and we can use the alternative definition (2.26) for the multifractal spectrum maps (with (f_t) the centered Loewner maps for η and $\psi = f_t^{-1}$).

Next, we note that $D \setminus \eta^t$ a.s. has only countably many connected components for any given $t > 0$. By continuity, countable stability of Hausdorff dimension, and Lemma 2.13, it therefore suffices to prove that the statement of the proposition holds a.s. for some fixed but arbitrary choice of domain V as in the statement of the proposition (chosen in some deterministic manner), rather than for all such V simultaneously (it will be clear from the proof that the a.s. dimension obtained does not depend on how we choose V).

We will prove the result for $\Theta^s(V) \setminus \mathbf{R}$ and $\tilde{\Theta}^s(V) \setminus f_t^{-1}(\mathbf{R})$; the case for the other sets is similar.

First consider the case where $\kappa \leq 4$ and $\underline{\rho} = 0$, so η is an ordinary SLE_κ process. In this case, the statement of the proposition for a complementary connected component V of $\mathbf{H} \setminus \eta$ follows from the statement for $V = \mathbf{H} \setminus \eta^t$ by Lemma 2.13 and countable stability of Hausdorff dimension, so it suffices to prove the statement with $V = \mathbf{H} \setminus \eta^t$ for a general choice of $t > 0$.

By scale invariance the law of each $\Theta^s(\mathbf{H} \setminus \eta^t)$ is independent of t . Since the derivative of the conformal map $f_{t/2}$ is bounded above and below by positive (random) constants in a neighborhood of each point of $\eta^t \setminus \eta^{t/2}$, we infer that $\Theta^s(\mathbf{H} \setminus \eta^t) \setminus \eta^{t/2} = \Theta^s(\mathbf{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2}))$.

Since conformal maps preserve Hausdorff dimension and by Lemma 2.13, we thus have that the Hausdorff dimension of each $\Theta^s(\mathbf{D} \setminus \eta^t)$ is equal to the maximum of $\dim_{\mathcal{H}} \Theta^s(\mathbf{H} \setminus \eta^{t/2})$ and $\dim_{\mathcal{H}} \Theta^s(\mathbf{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2}))$. These latter two sets are independent and identically distributed (by the Markov property of SLE) and their Hausdorff dimensions agree in law with that of $\Theta^s(\mathbf{H} \setminus \eta^t)$ (by the scale invariance property noted above). A random variable can be equal to the maximum of two independent random variables with the same law as itself only if it is a.s. constant.

To prove the analogous statement for $\tilde{\Theta}^s(\mathbf{H} \setminus \eta^t)$, we observe that $\dim_{\mathcal{H}} \tilde{\Theta}^s(\mathbf{H} \setminus \eta^t)$ is the maximum of $\dim_{\mathcal{H}} f_t^{-1}(\tilde{\Theta}^s(\mathbf{H} \setminus \eta^t) \cap \eta^{t/2})$ and $\dim_{\mathcal{H}} f_t^{-1}(\Theta^s(\mathbf{H} \setminus \eta^t) \setminus \eta^{t/2})$. By smoothness of the map $f_{t/2} \circ f_t^{-1}$ on $f_{t/2}(\mathbf{H} \setminus \eta^{t/2})$ and of f_t^{-1} on $\eta^t \setminus \eta^{t/2}$, respectively, these dimensions equal $\dim_{\mathcal{H}} f_{t/2}^{-1}(\tilde{\Theta}^s(\mathbf{H} \setminus \eta^{t/2}))$ and $\dim_{\mathcal{H}} (f_t \circ f_{t/2}^{-1})^{-1}(\tilde{\Theta}^s(\mathbf{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2})))$, respectively. By the Markov property these latter two quantities are i.i.d., and we conclude as above.

Next suppose that κ is still ≤ 4 , but that $\underline{\rho}$ is arbitrary. For $\delta > 0$, inductively define stopping times τ_j^δ and σ_j^δ for $j \in \mathbf{N}$ as follow. Let τ_1^δ be the first time $t > 0$ that either η hits an interior force point or the continuation threshold; or $\text{Im } \eta(t) \geq 2\delta$. Let σ_1^δ be the first time $s > \tau_1^\delta$ that η hits an interior force point or the continuation threshold; or $\text{Im } \eta(s) \leq \delta$. Inductively, if $j \geq 2$ and $(\tau_{j-1}^\delta, \sigma_{j-1}^\delta)$ has been defined, let τ_j^δ be the first time $t > \sigma_{j-1}^\delta$ that η either hits an interior force point or the continuation threshold; or $\text{Im } \eta(t) \geq 2\delta$ and let σ_j^δ be the first time $s > \tau_j^\delta$ that either η hits the continuation threshold or $\text{Im } \eta(s) \leq \delta$. For each δ and each j , the law of $f_{\tau_j^\delta}(\eta|_{[\tau_j^\delta, \sigma_j^\delta]})$ is absolutely continuous with respect to the law of an ordinary SLE_κ stopped at some a.s. positive time. Therefore, Lemma 2.13 implies that if V is as in the statement of the lemma and $\psi : \mathbf{H} \rightarrow V$ is a conformal map, then $\dim_{\mathcal{H}} \Theta^s(V) \cap \eta([\tau_j^\delta, \sigma_j^\delta])$ and $\dim_{\mathcal{H}} \tilde{\Theta}^s(V) \cap \psi^{-1}(\eta([\tau_j^\delta, \sigma_j^\delta]))$, respectively, a.s. agree with the a.s. values of the corresponding sets for an ordinary SLE_κ . We have

$$\bigcup_{\delta > 0} \bigcup_{j \in \mathbf{N}} \eta([\tau_j^\delta, \sigma_j^\delta]) = \eta \setminus \mathbf{R}$$

so by countable stability of Hausdorff dimension (restrict δ to a sequence tending to 0) and we obtain the statement of the proposition for a general $\text{SLE}_\kappa(\underline{\rho})$ process with $\kappa \leq 4$.

The statement for $\kappa > 4$ follows from the statement for $16/\kappa < 4$ together with Lemma 2.13 and SLE duality (see, e.g. [Zha08a, Zha10, Dub09a, MS16a, MS13]). \square

For the proof of Corollary 1.8, we will also need the analogue of Proposition 2.16 for the integral means spectrum.

Proposition 2.17. *Suppose we are in the setting of Proposition 2.16. Fix $a \in \mathbf{R}$. Almost surely, the following is true. Let V be a complementary connected component of either $D \setminus \eta$ or of $D \setminus \eta^t$ for any $t > 0$. Then $\text{IMS}_V^{\text{bulk}}(a)$ is equal to a deterministic constant which depends only on κ and a . This deterministic constant is the same if we replace κ with $16/\kappa$.*

Proof. The is proven similarly to Proposition 2.16 but with Lemma 2.15 used in place of Lemma 2.13. \square

3 One point estimates for the inverse maps

In this section we will prove derivative estimates for the inverse centered Loewner maps of a chordal SLE_κ process, which we state just below. Let $\kappa \in (0, 4]$. Let η be a chordal SLE_κ process from 0 to ∞ in \mathbf{H} . Let (f_t) be its centered Loewner maps. For $z \in \mathbf{H}$ with $\text{Im } z = \epsilon$, $u > 0$, $s \in (-1, 1]$, $c > 0$, and $r > 0$, let $\underline{E}^{s;u}(z; t) = \underline{E}^{s;u}(z; t, c, r)$ be the event that

$$c^{-1}\epsilon^{-s+u} \leq |(f_t^{-1})'(z)| \leq c\epsilon^{-s-u} \quad \text{and} \quad \text{Im } f_t^{-1}(z) \geq r. \quad (3.1)$$

Theorem 3.1. *Let $\kappa \in (0, 4]$. Let (f_t) be the centered Loewner maps of a chordal SLE_κ process from 0 to ∞ in \mathbf{H} . Let $z \in \mathbf{H}$ with $\text{Im } z = \epsilon \in (0, 1)$ and $R^{-1} \leq |\text{Re } z| \leq R$ for some $R > 1$. Define the event $\underline{E}^{s;u}(z; t) = \underline{E}^{s;u}(z; t, c, r)$ as above. Let $G(f_t, \mu)$ be the event of Definition 2.3. Let*

$$\alpha(s) = \frac{(4 + \kappa)^2 s^2}{8\kappa(1 + s)}, \quad \alpha_0(s) = \frac{(4 + \kappa)^2 s(2 + s)}{8\kappa(1 + s)^2}. \quad (3.2)$$

For each $t, c, r > 0$, each $\mu \in \mathcal{M}$, each $s \in (-1, 1]$, and each $R > 1$, we have

$$\mathbf{P}(\underline{E}^{s;u}(z; t) \cap G(f_t, \mu)) \preceq \epsilon^{\alpha(s) - \alpha_0(s)u}. \quad (3.3)$$

Furthermore, for each $r > 0$, there exists $t_* > 0$, such that for each $t \geq t_*$, we can find $\mu \in \mathcal{M}$ such that for each $c > 0$ and each $u > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$, we have

$$\mathbf{P}(\underline{E}^{s;u}(z; t) \cap G(f_t, \mu)) \succeq \epsilon^{\alpha(s) + \alpha_0(s)u}. \quad (3.4)$$

In both (3.3) and (3.4), the implicit constants in \preceq and \succeq depend on the other parameters but not on ϵ , and are uniform for $z \in \mathbf{H}$ with $R^{-1} \leq |\text{Re } z| \leq R$.

Remark 3.2. The reason for the condition $\text{Im } f_t^{-1}(z) \geq r$ in the definition of the event $\underline{E}^{s;u}(z; t)$ is because we are interested in the bulk of the curve, not the behavior near the starting point, so we want to eliminate contributions to $\mathbf{P}(c^{-1}\epsilon^{-s+u} \leq |(f_t^{-1})'(z)| \leq c\epsilon^{-s-u})$ coming from the event that $f_t^{-1}(z)$ is near 0. There are several reasons why we include the condition $G(f_t, \mu)$ in Theorem 3.1. One reason is that this condition implies an upper bound on the diameter of η^t and a bound on the distance from η to the boundary once η has left a small neighborhood of the origin (see Lemma 2.4), which is needed to estimate some the auxiliary terms which arise in our proof. Furthermore, in the sequel, we will often normalize our conformal maps by specifying the images of three marked points on the boundary. The condition $G(\cdot, \mu)$ is needed so that the derivative of a conformal automorphism which takes the images of these marked points to where we want them to be is not too large or too small. Another use for the condition $G(\cdot, \mu)$ (and its analogue for the disk, $\mathcal{G}(\cdot, \mu)$) is in checking the hypotheses of the lemmas in Section B (see in particular Remark B.2).

Remark 3.3. Estimates similar to Theorem 3.1 can be deduced in a somewhat more efficient manner from the results in [RS05, Section 3] and those of [BS09]. In particular, [RS05, Lemma 3.3] implies the upper bound (3.3) for a restricted range of parameter values and an estimate similar to (3.4) can be deduced from [RS05, Corollary 3.5]. Additionally, a version of Theorem 3.1 for whole-plane SLE can be obtained

using the moment estimates of [BS09]. These estimates lead to a.s. upper bounds for the integral means spectrum of SLE and for the dimension of the set $\tilde{\Theta}^s(D_\eta) \subset \partial\mathbf{D}$ (at least for certain parameter values) via arguments similar to those given in Section 5.1 and 5.3. However, these results do not include the additional regularity conditions on the event in the lower bound of Theorem 3.1, so do not lead to proofs of the lower bounds in Theorem 1.1 and Corollary 1.8. Most of the work in the proof of Theorem 3.1 comes from obtaining a lower bound with these regularity conditions.

3.1 Reverse SLE martingales and upper bound

Let (g_t) be the centered Loewner maps of a reverse SLE_κ flow, so

$$dg_t(z) = -\frac{2}{g_t(z)} dt - dW_t, \quad g_0(z) = z \quad (3.5)$$

for $W_t = \sqrt{\kappa}B_t$ and (B_t) a standard linear Brownian motion. Our interest in (g_t) stems from the fact that if (f_t) is as in Theorem 3.1, then $g_t \stackrel{d}{=} f_t^{-1}$ for each t (see, e.g. [RS05, Lemma 3.1]).

Let $K_t = \mathbf{H} \setminus g_t(\mathbf{H})$ be the hulls corresponding to (g_t) . Since $f_t^{-1} \stackrel{d}{=} g_t$ for each t , it is only a minor abuse of notation to replace f_t^{-1} with g_t in the definition of the events of Theorem 3.1, and we do so in the remainder of this section.

3.1.1 Reverse SLE martingales

We state here a result originally due to Lawler [Law09, Proposition 2.1], but in a form which is more convenient for our purposes.

Lemma 3.4. *Let $\kappa > 0$. Let (g_t) be as above, $\rho \in \mathbf{R}$, $z \in \mathbf{H}$, and*

$$M_t^z = |g_t'(z)|^{\frac{(8+2\kappa-\rho)\rho}{8\kappa}} (\text{Im } g_t(z))^{-\frac{\rho^2}{8\kappa}} |g_t(z)|^{\rho/\kappa}. \quad (3.6)$$

Then M_t^z is a martingale. Let \mathbf{P}_^z be the law of (g_t) weighted by M^z . The law of (g_t) under \mathbf{P}_*^z is that of the centered Loewner maps of a reverse $\text{SLE}_\kappa(\rho)$ with a force point at z . That is, under the reweighted law,*

$$dW_t = -\text{Re} \frac{\rho}{g_t(z)} dt + \sqrt{\kappa} dB_t^z \quad (3.7)$$

for B_t^z a \mathbf{P}_^z -Brownian motion.*

Remark 3.5. The martingale (3.6) is the reverse SLE analogue of the local martingale of [SW05, Section 5] in the case of a single force point.

3.1.2 Proof of the upper bound

In this subsection we will prove (3.3) of Theorem 3.1. We will actually prove something a little stronger, namely the following.

Proposition 3.6. *Let $\kappa > 0$. Let $\alpha(s)$ be as in (3.2) and let (g_t) be the centered Loewner maps of a reverse SLE_κ as above. Let $c, t, d > 0$. For $s \in [0, 1]$ and $z \in \mathbf{H}$ with $\text{Im } z = \epsilon \in (0, 1)$, let $\underline{E}^{s;\infty}(z; t) = \underline{E}^{s;\infty}(z; t, c, d)$ be the event that $|g_t'(z)| \geq c^{-1}\epsilon^{-s}$ and $|g_t(z)| \geq d^{-1}$. For $s \in (-1, 0)$, let $\underline{E}^{s;\infty}(z; t) = \underline{E}^{s;\infty}(z; t, c, d)$ be the event that $|g_t'(z)| \leq c\epsilon^{-s}$ and $|g_t(z)| \leq d$. For any bounded stopping time τ for (g_t) we have*

$$\mathbf{P}(\underline{E}^{s;\infty}(z; \tau)) \preceq \epsilon^{\alpha(s)}. \quad (3.8)$$

For any $R > 1$, the implicit constant in (3.8) is uniform for $z \in \mathbf{H}$ with $R^{-1} \leq |\text{Re } z| \leq R$.

The estimate (3.3) is immediate from Proposition 3.6 in the case $s \in [0, 1]$. To extract (3.3) from Proposition 3.6 in the case $s \in (-1, 0)$, we observe that Lemma 2.4 implies that $\text{diam } K_t$ is bounded by a constant depending only on t and μ on the event $G(g_t^{-1}, \mu)$ (c.f. the discussion following Definition 2.3). For $R^{-1} \leq |\text{Re } z| \leq R$, [Law05, eqn. 3.14] then implies that $|g_t(z)|$ is bounded by a constant depending only on t, μ , and R on $\underline{E}^{s;u}(z; t) \cap G(g_t^{-1}, \mu)$. Thus we have $\underline{E}^{s;u}(z; t) \cap G(g_t^{-1}, \mu) \subset \underline{E}^{s+u;\infty}(z; t, c, d)$ for a suitable choice of d .

Proof of Proposition 3.6. Throughout, we fix $R > 1$ and require all implicit constants to be uniform for $z \in \mathbf{H}$ with $R^{-1} \leq |\operatorname{Re} z| \leq R$. Let

$$\rho = \rho(s) := \frac{(4 + \kappa)s}{1 + s}. \quad (3.9)$$

and denote by \mathbf{P}_*^z the law of (g_t) re-weighted by the martingale of Lemma 3.4 with this choice of ρ . By the Loewner equation, $\operatorname{Im} g_\tau(z)$ is bounded above by a constant depending only on the essential supremum of τ . Therefore,

$$M_\tau^z \mathbf{1}_{\underline{E}^{s;\infty}(z;\tau)} \succeq \epsilon^{\frac{-s(8+2\kappa-\rho)\rho}{8\kappa}} \mathbf{1}_{\underline{E}^{s;\infty}(z;\tau)} \quad (3.10)$$

(we can replace the \succeq with an \asymp if we assume that $\operatorname{Im} g_t(z)$ is bounded below and $|g_t(z)|$ is bounded above). Furthermore, if $R^{-1} \leq |\operatorname{Re} z| \leq R$ then

$$M_0^z \asymp \epsilon^{-\frac{\rho^2}{8\kappa}}. \quad (3.11)$$

Thus the optional stopping theorem implies

$$\epsilon^{\frac{-s(8+2\kappa-\rho)\rho}{8\kappa}} \mathbf{P}(\underline{E}^{s;\infty}(z;\tau)) \asymp \mathbf{E}(M_\tau^z \mathbf{1}_{\underline{E}^{s;\infty}(z;\tau)}) \leq \epsilon^{-\rho^2/8\kappa} \mathbf{P}_*^z(\underline{E}^{s;\infty}(z;\tau)).$$

Therefore

$$\mathbf{P}(\underline{E}^{s;\infty}(z;\tau)) \leq \epsilon^{\frac{s(8+2\kappa-\rho)\rho}{8\kappa} - \frac{\rho^2}{8\kappa}} \mathbf{P}_*^z(\underline{E}^{s;\infty}(z;\tau)). \quad (3.12)$$

The value of the exponent on the right is maximized by taking $\rho = \rho(s)$, as in (3.9). Choosing this value of ρ yields the upper bound (3.8). \square

3.2 Reduction of the lower bound to a result for a stopping time

Now we turn our attention to the lower bound (3.4) in Theorem 3.1. We continue to assume that we have replaced f_t^{-1} with g_t in the definition of the events of Theorem 3.1, as in Section 3.1.

Let T_r^z be the first time t that $\operatorname{Im} g_t(z) \geq r$ and fix a time $\bar{t} > 0$. Put

$$\tau = \tau_r^z := T_r^z \wedge \bar{t}, \quad (3.13)$$

so that up to an event of probability zero, we have

$$\{\tau < \bar{t}\} = \{\operatorname{Im} g_\tau(z) \geq r\} = \{\operatorname{Im} g_{\bar{t}}(z) \geq r\}.$$

We claim that to prove that (3.4) holds at time \bar{t} , and hence to finish the proof of Theorem 3.1, it is enough to prove the following statement.

Proposition 3.7. *Let $\rho = \rho(s)$ be as in (3.9). Let \mathbf{P}_*^z be the law of a reverse $\operatorname{SLE}_\kappa(\rho)$ process (g_t) with hulls (K_t) , with an interior force point located at $z \in \mathbf{H}$ with $\operatorname{Im} z = \epsilon$. Let $\tau = \tau_r^z$ be as in (3.13). Define the events $\underline{E}^{s;u}(z;\tau_r^z)$ as in (3.1), but with (g_t) in place of (f_t) and the time τ hull K_τ for (g_t) in place of η^τ . For each $R > 1$ there exists $r_* > 0$ such that for each $r \geq r_*$, we can find $\mu \in \mathcal{M}$ and $t_* > 0$ such that for each $u > 0$ there exists $\epsilon_0 > 0$ such that for each $z \in \mathbf{H}$ with $\operatorname{Im} z = \epsilon \leq \epsilon_0$ and $R^{-1} \leq |\operatorname{Re} z| \leq R$ and each $\bar{t} \geq t_*$, we have*

$$\mathbf{P}_*^z(\underline{E}^{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)) \succeq 1. \quad (3.14)$$

Here the implicit constant and all of the above parameters are independent of ϵ and uniform for z with $R^{-1} \leq |\operatorname{Re} z| \leq R$ (but may depend on r, R, μ, \bar{t} , and u).

We will prove Proposition 3.7 in the subsequent subsections. In the remainder of this subsection we deduce Theorem 3.1 from Proposition 3.7. To lighten notation, in what follows we write $\tau = \tau_r^z$.

First we note that the probability of the event of Theorem 3.1 is decreasing in r , so it suffices to prove (3.4) for $r \geq r_*$, with r_* as in Proposition 3.7. Observe that $|g_\tau(z)|$ is a.s. bounded above by a positive constant on the event $\underline{E}^{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)$ (c.f. Section 3.1). By combining this with the definition of $\underline{E}^{s;u}(z;\tau)$ we see that

$$M_\tau^z \mathbf{1}_{\underline{E}^{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)} \leq \epsilon^{\frac{-(s+u)(8+2\kappa-\rho)\rho}{8\kappa}} \mathbf{1}_{\underline{E}^{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)}.$$

By (3.11) and our choice (3.9) of ρ we then have

$$\epsilon^{\alpha(s)+\alpha_0(s)u} \mathbf{P}_*^z(\underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)) \preceq \mathbf{P}(\underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)). \quad (3.15)$$

Assuming that Proposition 3.7 holds, (3.15) implies (3.4) with τ in place of t . To get the desired bound at the deterministic time \bar{t} , for $t \geq \tau$ let $g_{\tau,t}$ be the conformal map defined on \mathbf{H} which satisfies $g_{\tau,t} \circ g_\tau = g_t$. By the strong Markov property the conditional law given $\{g_t : t \leq \tau\}$ of the family of conformal maps $\{g_{\tau,v+\tau} : v \geq 0\}$ is the same as the law of the $\{g_v : v \geq 0\}$. For $w \in \mathbf{C}$, $\mu' \in \mathcal{M}$ and $C > 1$, let $E_{\tau,\bar{t}}(w) = E_{\tau,\bar{t}}(w; C, \mu')$ be the event that the following is true.

1. $C^{-1} \leq |g'_{\tau,t}(w)| \leq C$ for each $t \in [\tau, \bar{t}]$.
2. $G(g_{\tau,t}^{-1}, \mu')$ occurs for each $t \in [\tau, \bar{t}]$.

If C is chosen sufficiently large and $\mu' \in \mathcal{M}$ is chosen sufficiently small, depending on \bar{t} but uniform for w in compact subsets of \mathbf{H} , then $\mathbf{P}(E_{\tau,\bar{t}}(w))$ is at least a positive constant depending uniformly on w in compact subsets of \mathbf{H} . Furthermore, since we have a bound on $\text{diam } K_\tau$ on the event $\underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)$ (see Lemma 2.4), it follows from the Markov property that

$$\mathbf{P}(E_{\tau,\bar{t}}(g_\tau(z)) \cap \underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)) \succeq \mathbf{P}(\underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)).$$

On the other hand, the definition of $E_{\tau,\bar{t}}(g_\tau(z))$ implies that

$$E_{\tau,\bar{t}}(g_\tau(z)) \cap \underline{E}^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu) \subset \underline{E}^{s;u}(z; \bar{t}, c', r) \cap G(g_\tau^{-1}, \mu \circ \mu')$$

for some $c' > 0$ depending on the other parameters (here we use that $\text{Im } g_t(z)$ is increasing in t for the condition involving r). By making c sufficiently small, we can make c' as small as we like. We conclude that (3.4) at time τ implies (3.4) at time \bar{t} .

Thus to prove Theorem 3.1 it remains to prove Proposition 3.7. The proof is separated into two major steps: first we prove that the derivative condition in the definition of $\underline{E}^{s;u}(z)$ holds at time τ with \mathbf{P}_*^z -probability tending to 1 as $\epsilon = \text{Im } z \rightarrow 0$. This is done in Section 3.3 via a coupling with a Gaussian free field. Then we prove that $\mathbf{P}_*^z(\{\tau < \bar{t}\} \cap G(g_\tau^{-1}, \mu))$ is uniformly positive for sufficiently small μ and sufficiently large \bar{t} . This is done in Appendix A via a stochastic calculus argument.

3.3 Derivative estimate via reverse SLE/GFF coupling

Assume we are in the setting of Proposition 3.7. In this subsection we will prove that $\text{Im } |g'_\tau(z)| \approx \epsilon^{-s}$ with high probability under \mathbf{P}_*^z . We do this using a coupling with a Gaussian free field.

Let h be a free boundary GFF on \mathbf{H} , independent from (g_t) , normalized so that its harmonic part \mathfrak{h} vanishes at iy for some $y > 0$ (which we will specify below in such a way that it depends on \bar{t} , but not ϵ). Let \mathbf{P}_h be the law of h . For $t \geq 0$ let

$$h_t = h \circ g_t + \frac{2}{\sqrt{\kappa}} \log |g_t(\cdot)| + \frac{\rho}{2\sqrt{\kappa}} G(g_t(z), g_t(\cdot)), \quad (3.16)$$

where

$$G(x, y) := -\log |x - y| - \log |\bar{x} - y|$$

is the Green's function on \mathbf{H} with Neumann boundary conditions.

Let τ be as in (3.13). By [She16, Theorem 2.5], we have $h_\tau + Q \log |g'_\tau| \stackrel{d}{=} h_0$, modulo additive constant, where $Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}$ is as in (2.14). Let b_τ be this additive constant, so

$$h_\tau + Q \log |g'_\tau| - b_\tau \stackrel{d}{=} h_0. \quad (3.17)$$

The idea of the proof of (3.7) is to estimate the terms other than $\log |g'_\tau|$ in (3.17), and thereby obtain an estimate for $|g'_\tau|$. See the proof of [MS16d, Theorem 8.1] for another argument using a similar idea.

Let

$$\tilde{h}_0 = h_\tau + Q \log |g'_\tau| - b_\tau \quad (3.18)$$

so that by (3.17) we have $\tilde{h}_0 \stackrel{d}{=} h_0$. Rearranging the definition of \tilde{h}_0 gives

$$\begin{aligned} Q \log |g'_\tau(w)| &= \tilde{h}_0 - h_\tau + b_\tau \\ &= \tilde{h} - h \circ g_\tau + \frac{2}{\sqrt{\kappa}} \log \frac{|w|}{|g_\tau(w)|} + \frac{\rho}{2\sqrt{\kappa}} \left(\log \frac{|g_\tau(w) - g_\tau(z)|}{|w - z|} + \log \frac{|g_\tau(w) - \overline{g_\tau(z)}|}{|w - \bar{z}|} \right) + b_\tau, \end{aligned} \quad (3.19)$$

where here \tilde{h} is a field with the same law as h and we use w instead of \cdot as a dummy variable. Since all of the non-GFF terms in (3.19) are harmonic away from z , the equation still holds for $w \neq z$ if we replace \tilde{h} and $h \circ g_\tau$ with the circle average processes \tilde{h}_ϵ and $(h \circ g_\tau)_\epsilon$ for these two fields. We will use (3.19) to estimate b_τ and then to estimate $|g'_\tau(z)|$.

Throughout the rest of this subsection, we fix $R > 1$, $c > 0$, $r > 0$, $\mu \in \mathcal{M}$, and $\bar{t} > 0$ and require all implicit constants to be independent of ϵ and uniform for $R^{-1} \leq |\operatorname{Re} z| \leq R$ and all $o_\epsilon(1)$ errors to be uniform for $R^{-1} \leq |\operatorname{Re} z| \leq R$. These quantities are, however, allowed to depend on R , c , r , μ , \bar{t} , s , and u .

Lemma 3.8. *Let $\xi > 1/2$. If y is chosen sufficiently large (independently of ϵ and uniform for $R^{-1} \leq |\operatorname{Re} z| \leq R$) then*

$$(\mathbf{P}_*^z \otimes \mathbf{P}_h) \left(\{|b_\tau| > (\log \epsilon^{-1})^\xi\} \cap G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\} \right) = o_\epsilon(1). \quad (3.20)$$

Proof. If we replace the GFF terms with circle averages in (3.19) and evaluate at $w = iy$, we get

$$\begin{aligned} Q \log |g'_\tau(iy)| &= \tilde{h}_\epsilon(iy) - (h \circ g_\tau)_\epsilon(iy) + \frac{2}{\sqrt{\kappa}} \log \frac{y}{|g_\tau(iy)|} \\ &\quad + \frac{\rho}{2\sqrt{\kappa}} \left(\log \frac{|g_\tau(iy) - g_\tau(z)|}{|iy - z|} + \log \frac{|g_\tau(iy) - \overline{g_\tau(z)}|}{|iy - \bar{z}|} \right) + b_\tau. \end{aligned} \quad (3.21)$$

By Lemma 2.4 $\operatorname{diam} K_\tau \leq 1$ on $G(g_\tau^{-1}, \mu)$. By [Law05, Proposition 3.46] we have $\operatorname{Im} g_\tau(iy) \asymp |g_\tau(iy)| \asymp 1$ on $G(g_\tau^{-1}, \mu)$. By the Koebe quarter theorem we also have $|g'_\tau(iy)| \asymp 1$ on $G(g_\tau^{-1}, \mu)$ provided y is chosen sufficiently large, depending only on μ , \bar{t} , and R . Hence each of the terms in (3.21) except for b_τ and the two circle averages is $\asymp 1$ on $G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\}$ (implicit constants also depending on y) if y is chosen sufficiently large, depending only on μ , \bar{t} , and R . By Lemma 2.10, for $\xi > 1/2$ we have

$$(\mathbf{P}_*^z \otimes \mathbf{P}_h) \left(|\tilde{h}_\epsilon(iy) - (h \circ g_\tau)_\epsilon(iy)| > (\log \epsilon)^\xi \right) = o_\epsilon(1).$$

Note that we took $A = \emptyset$ in that lemma to estimate $\tilde{h}_\epsilon(iy)$ and we took $A = K_\tau$ and used that K_τ is independent of h to estimate $(h \circ g_\tau)_\epsilon(iy)$. By re-arranging (3.21) we conclude. \square

Proposition 3.9. *Suppose we define $\rho = \rho(s)$ as in (3.9). For any $c > 0$, we have*

$$\mathbf{P}_*^z \left(\{|g'_\tau(z)| \notin [c^{-1}\epsilon^{-s+u}, c\epsilon^{-s-u}]\} \cap G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\} \right) = o_\epsilon(1). \quad (3.22)$$

Proof. Since the circle average process is continuous [DS11, Proposition 3.1], we can take the limit as $w \rightarrow z$ in (3.19) to get

$$\begin{aligned} Q \log |g'_\tau(z)| &= \tilde{h}_\epsilon(z) - (h \circ g_\tau)_\epsilon(z) + \frac{\rho}{2\sqrt{\kappa}} \log |g'_\tau(z)| - \frac{\rho}{2\sqrt{\kappa}} \log \epsilon \\ &\quad + \frac{2}{\sqrt{\kappa}} \log \frac{|z|}{|g_\tau(z)|} + \frac{\rho}{2\sqrt{\kappa}} \log |\operatorname{Im} g_\tau(z)| + b_\tau. \end{aligned} \quad (3.23)$$

Since we have a uniform upper bound on $\operatorname{diam} K_\tau$ on the event $G(g_\tau^{-1}, \mu)$ and $\operatorname{Im} g_\tau(z) = r$ on the event $\{\tau < \bar{t}\}$, the absolute value of the sum of the fifth and sixth terms in the right in (3.23) is ≤ 1 on $G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\}$.

By Lemma 2.10 (applied as in the proof of Lemma 3.8), for any $\xi > 1/2$,

$$(\mathbf{P}_*^z \otimes \mathbf{P}_h) \left(|\tilde{h}_\epsilon(z) - (h \circ g_\tau)_\epsilon(z)| \geq (\log \epsilon^{-1})^\xi \right) = o_\epsilon(1).$$

By Lemma 3.8, the probability that the last term in (3.23) is $\geq (\log \epsilon)^{1/2}$ and $G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\}$ occurs is of order $o_\epsilon(1)$. Hence except on an event of $\mathbf{P}_*^z \otimes \mathbf{P}_h$ -probability of order $o_\epsilon(1)$, on the event $G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\}$ it holds that

$$Q \log |g'_\tau(z)| = \frac{\rho}{2\sqrt{\kappa}} \log |g'_\tau(z)| + \frac{\rho}{2\sqrt{\kappa}} \log \epsilon^{-1} + o_\epsilon(\log \epsilon^{-1}).$$

Rearranging, we get that except on an event of $\mathbf{P}_*^z \otimes \mathbf{P}_h$ -probability of order $o_\epsilon(1)$, on the event $G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\}$ we have

$$\log |g'_\tau(z)| = \frac{\rho}{\kappa + 4 - \rho} \log \epsilon^{-1} + o_\epsilon(\log \epsilon^{-1}). \quad (3.24)$$

With ρ as in (3.9) we have

$$\frac{\rho}{\kappa + 4 - \rho} = s,$$

so integrating out \mathbf{P}_h yields (3.22). \square

3.4 Proof of Proposition 3.7

In light of Proposition 3.9, to prove Proposition 3.7, and hence Theorem 3.1, it remains to prove that $\mathbf{P}_*^z (G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\})$ is uniformly positive. In particular, we will prove the following.

Proposition 3.10. *Let (g_t) be as in (3.5). and let (K_t) be the associated hulls. Let $z \in \mathbf{H}$. For $r > \text{Im } z$ let T_r^z be the first time t that $\text{Im } g_t(z) = r$. Let $\rho \in (-\infty, \kappa/2 + 2)$ and let \mathbf{P}_*^z be the law of (g_t) weighted by M^z , as in Lemma 3.4. For any given $R > 1$, there exists $r_* > 0$ such that for each $r \geq r_*$, we can find $\mu \in \mathcal{M}$, $t_* > 0$, $\epsilon_0 > 0$, and $p > 0$ such that for $z \in \mathbf{H}$ with $|\text{Re } z| \leq R$ and $\text{Im } z \leq \epsilon_0$, we have*

$$\mathbf{P}_*^z \left(\{T_r^z < t_*\} \cap G(g_{T_r^z}^{-1}, \mu) \right) \geq p. \quad (3.25)$$

The proof of Proposition 3.10 is given in Appendix A. In the remainder of this section, we use Proposition 3.10 to conclude the proof of Proposition 3.7, and hence (recall Section 3.2) the proof of Theorem 3.1.

Proof of Proposition 3.7. Fix $R > 1$ and $c > 0$. Let $r_* > 0$ be as in Proposition 3.10 for this choice of R . Given $r \geq r_*$, let $\mu \in \mathcal{M}$, $\bar{t} > 0$, $\epsilon_0 > 0$, and $p > 0$ be as in Proposition 3.10, so that (3.25) holds. Given $\bar{t} \geq t_*$, let τ be as in (3.13). By Proposition 3.9, we can find $\epsilon'_0 \in (0, \epsilon_0)$ (depending on c, R, \bar{t}, r, μ, s , and u) such that whenever $z \in \mathbf{H}$ with $R^{-1} \leq |\text{Re } z| \leq R$ and $\text{Im } z = \epsilon \in (0, \epsilon'_0)$, we have

$$\mathbf{P}_*^z \left(\{|g'_\tau(z)| \notin [c^{-1}\epsilon^{-s+u}, c\epsilon^{-s-u}]\} \cap G(g_\tau^{-1}, \mu) \cap \{\tau < \bar{t}\} \right) \leq p/2.$$

If $T_r^z < t_* \leq \bar{t}$, then $\tau < \bar{t}$ and $\text{Im } g_\tau(z) \geq r$. By (3.25), it follows that for such a choice of z we have

$$\mathbf{P}_*^z (E^{s;u}(z; \tau) \cap G(g_\tau^{-1}, \mu)) \geq p/2. \quad (3.26)$$

\square

3.5 Estimates for chordal SLE in the disk

In the sequel we will work mostly in the unit disk \mathbf{D} rather than in the upper half plane \mathbf{H} . In this brief subsection we make some trivial remarks about how Theorem 3.1 generalizes to this setting.

Suppose η is a chordal SLE $_\kappa$ from $-i$ to i in \mathbf{D} . Let $\psi : \mathbf{D} \rightarrow \mathbf{H}$ be the conformal map taking $-i$ to 0 , i to ∞ , and having positive real derivative at 0 . Suppose η is parametrized in such a way that $\psi(\eta)$ is parametrized by half-plane capacity. For each time $t \geq 0$, let

$$f_t : \mathbf{D} \setminus \eta^t \rightarrow \mathbf{D}$$

be defined so that $\psi \circ f_t \circ \psi^{-1}$ is the time t centered Loewner map for $\psi(\eta)$.

For $s \in (-1, 1)$, $u > 0$, $z \in \mathbf{D}$ with $1 - |z| = \epsilon$ and $t, c, d > 0$, let $\underline{E}_{\mathbf{D}}^{s;u}(z; t) = \underline{E}_{\mathbf{D}}^{s;u}(z; t, c, d)$ be the event that

$$\epsilon^{-s+u} \leq |(f_t^{-1})'(z)| \leq \epsilon^{-s-u} \quad \text{and} \quad f_t^{-1}(z) \in B_d(0).$$

Then in this context Theorem 3.1 reads as follows.

Corollary 3.11 (Theorem 3.1 for the disk). *Suppose we are in the setting described just above. Let $\delta > 0$ and let $z \in \mathbf{D}$ with $|z - i|, |z + i| \geq \delta$ and $1 - |z| = \epsilon$. Define the events $\mathcal{G}(\cdot)$ as in Definition 2.5. For each $t, c, d, \delta > 0$, each $s \in (-1, 1]$, and each $\mu \in \mathcal{M}$,*

$$\mathbf{P}(\underline{E}_{\mathbf{D}}^{s;u}(z; t) \cap \mathcal{G}(f_t, \mu)) \preceq \epsilon^{\alpha(s) - \alpha_0(s)u}. \quad (3.27)$$

Furthermore, there exists $t_* > 0$ such that for each $t \geq t_*$, we can find $\mu \in \mathcal{M}$ and $d \in (0, 1)$ such that for each $c > 0$ and each $u > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$,

$$\mathbf{P}(\underline{E}_{\mathbf{D}}^{s;u}(z; t) \cap \mathcal{G}(f_t, \mu)) \succeq \epsilon^{\alpha(s) + \alpha_0(s)u}. \quad (3.28)$$

In both (3.27) and (3.28), the implicit constants in \preceq and \succeq depend on the other parameters but not on ϵ , and are uniform for $z \in \mathbf{D}$ with $|z - i|, |z + i| \geq \delta$.

Proof. This is immediate from Theorem 3.1 and a coordinate change. Note that we use Lemma 2.4 to obtain a $d \in (0, 1)$, depending on μ , such that (3.28) holds. \square

4 One point estimates for the forward maps

4.1 Statement of the estimates

In this section we transfer the estimates of Theorem 3.1 to estimates for certain “time infinity” forward Loewner maps, which we will define shortly. We work in the setting of \mathbf{D} , rather than \mathbf{H} , as this setting will be more convenient for our two-point estimates. We start by defining our events.

Let $x, y \in \partial\mathbf{D}$ be distinct and let m be the midpoint of the counterclockwise arc connecting x and y in $\partial\mathbf{D}$. Suppose we are given a simple curve η in \mathbf{D} connecting x and y . Let D_η be the connected component of $\mathbf{D} \setminus \eta$ containing m on its boundary. Let $\Psi_\eta : D_\eta \rightarrow \mathbf{D}$ be the unique conformal map taking x to $-i$, y to i , and m to 1. For $s \in \mathbf{R}$, $u > 0$, $\epsilon > 0$, $c > 1$, and $z \in \mathbf{D}$, let $\mathcal{E}_\epsilon^{s;u}(\eta, z; c)$ be the event that

1. $z \in D_\eta$;
2. $c^{-1}\epsilon^{1-s+u} \leq \text{dist}(z, \partial D_\eta) \leq c\epsilon^{1-s-u}$; and
3. $c^{-1}\epsilon^{s+u} \leq |\Psi'_\eta(z)| \leq c\epsilon^{s-u}$.

For technical reasons it will also be convenient to consider the counterclockwise arc of $\partial\mathbf{D}$ from y to x . We denote by m^- the midpoint of this arc. Let D_η^- be the connected component of $\mathbf{D} \setminus \eta$ containing m^- on its boundary and we let $\Psi_\eta^- : D_\eta^- \rightarrow \mathbf{D}$ be the unique conformal map taking x to i , taking y to $-i$, and taking m^- to -1 . See Figure 4.1 for an illustration.

Let $\mathcal{A}_\epsilon^{s;u}(\eta, c)$ be the set of $z \in \mathbf{D}$ for which $\mathcal{E}_\epsilon^{s;u}(\eta, z; c)$ occurs.

Theorem 4.1. *Suppose we are in the setting described just above. Let $\alpha(s)$ and $\alpha_0(s)$ be as in (3.2) and define*

$$\gamma(s) := \alpha(s) - 2s + 1 = \frac{(4 + \kappa)^2 s^2}{8\kappa(1 + s)} - 2s + 1, \quad \gamma_0(s) := 2\alpha_0(s) + 2 = \frac{2(4 + \kappa)^2 s(2 + s)}{8\kappa(1 + s)^2} + 2. \quad (4.1)$$

Also define the events $\mathcal{G}(\cdot, \mu)$ as in Definition 2.5. For each $d \in (0, 1)$, $\mu \in \mathcal{M}$, $c > 0$, and $z \in B_d(0)$, we have

$$\mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) \preceq \epsilon^{\gamma(s) - \gamma_0(s)u}. \quad (4.2)$$

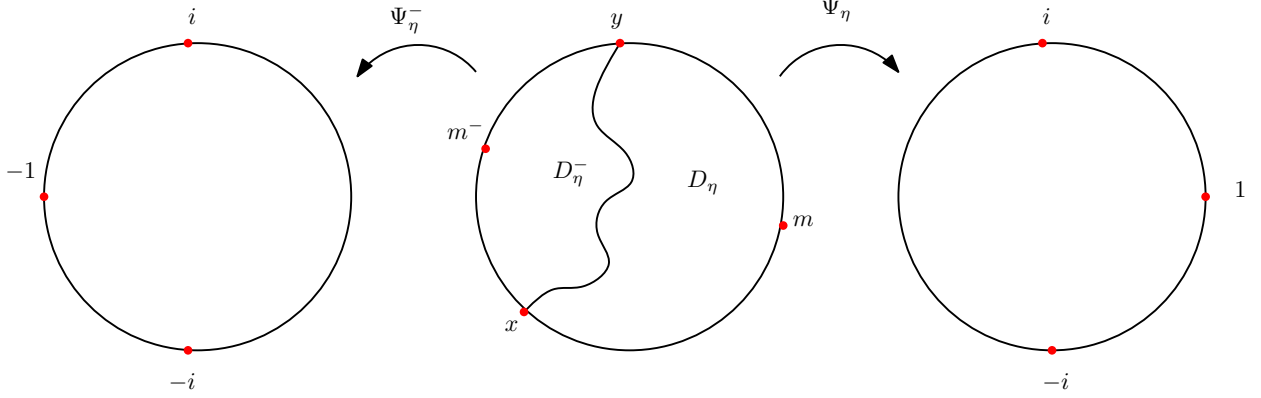


Figure 4.1: An illustration of the domains and maps used in Theorem 4.1.

Furthermore, for each $d \in (0, 1)$ there exists $\mu \in \mathcal{M}$ such that for each $c > 0$ and $u > 0$ we can find $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$ and each $z \in B_d(0)$,

$$\mathbf{P} \left(\mathcal{E}_\epsilon^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu) \right) \succeq \epsilon^{\gamma(s)+2\gamma_0(s)u}. \quad (4.3)$$

In (4.2) and (4.3) the implicit constants are independent of ϵ and uniform for $z \in B_d(0)$ and for $|x - y|$ bounded below by a positive constant.

The proof of Theorem 4.1 proceeds as follows. First we use Theorem 3.1 to prove estimates for the area of certain finite-time analogues of the sets of Theorem 4.1. This is done in Section 4.2. This subsection also contains a result which allows us to extend the estimate for deterministic times to estimates for certain stopping times, which will be needed in the sequel. Then, in Section 4.3, we prove several lemmas comparing finite time and infinite time maps and use these lemmas to obtain estimates for the area of the set $\mathcal{A}_\epsilon^{s;u}(\eta; c)$ of points where the events of Theorem 4.1 occur. Finally, we complete the proof of Theorem 4.1 in Section 4.4 by proving a lemma which gives that the probabilities of the events of Theorem 4.1 do not depend too strongly on z . In Section 4.5 we deduce an analogue of Theorem 4.1 for the curve stopped at a finite time.

4.2 Area estimates and stopping estimates for finite time maps

In this section we will prove estimates for the expected area of the set of points where finite-time analogues of the events of Theorem 4.1 occur. We will also prove a result which allows us to compare probabilities for events at stopping times whose difference is bounded. Suppose we are in the setting of Theorem 4.1.

Definition 4.2. Let η be a chordal SLE_κ from $-i$ to i in \mathbf{D} . Define its centered Loewner maps (f_t) as in Section 3.5. For $t, \epsilon, u, \delta, c > 0$, $s \in (-1, 1)$, and $z \in \mathbf{D}$, let $E_\epsilon^{s;u}(\eta, z; t, \delta, c)$ be the event that the following is true.

1. $c^{-1}\epsilon^{s+u} \leq |f'_t(z)| \leq c\epsilon^{s-u}$.
2. $c^{-1}\epsilon^{1-s+u} \leq \text{dist}(z, \eta^t) \leq c\epsilon^{1-s-u}$.
3. $|f_t(z) - i|$ and $|f_t(z) + i|$ are both at least δ .

Let $A_\epsilon^{s;u}(\eta; t, \delta, c)$ be the set of $z \in \mathbf{D}$ for which $E_\epsilon^{s;u}(\eta, z; t, \delta, c)$ occurs.

Lemma 4.3. Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. Fix $\delta > 0$. Define the sets $A_\epsilon^{s;u}(\eta; t, \delta, c)$ as in Definition 4.2 and the events $\mathcal{G}(f_t, \mu)$ as in Definition 2.5. For any choice of parameters t, c, μ and any $d \in (0, 1)$,

$$\mathbf{E} \left[\text{Area}(A_\epsilon^{s;u}(\eta; t, \delta, c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(f_t, \mu)} \right] \preceq \epsilon^{\gamma(s)-\gamma_0(s)u} \quad (4.4)$$

with the implicit constants independent of ϵ and uniform for $z \in B_d(0)$. Moreover, there exists $t_* > 0$ such that for each $t \geq t_*$, there exists $\mu \in \mathcal{M}$ and $d \in (0, 1)$ such that for each $c > 0$ and each $u > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$,

$$\mathbf{E} [\text{Area}(A_\epsilon^{s;u}(\eta; t, \delta, c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(f_t, \mu)}] \succeq \epsilon^{\gamma(s) + \gamma_0(s)u}, \quad (4.5)$$

with the implicit constants independent of ϵ and uniform for $z \in B_d(0)$.

Proof. Let $\underline{A}_\epsilon^{s;u} = \underline{A}_\epsilon^{s;u}(\eta; t, \delta, c, d)$ be the set of $z \in \mathbf{D}$ such that

1. $c^{-1}\epsilon^{1+u} \leq 1 - |z| \leq c\epsilon^{1-u}$;
2. $|z - i|$ and $|z + i|$ are each at least δ ;
3. The event $\underline{E}_\mathbf{D}^{s;u}(z; t, c, d)$ of Section 3.5 occurs.

By (3.27) in Theorem 3.11, if the first two conditions in the definition of $\underline{A}_\epsilon^{s;u}$ hold for some $z \in \mathbf{D}$, then

$$\mathbf{P}(\underline{E}_\mathbf{D}^{s;u}(z; t, c, d) \cap \mathcal{G}(f_t, \mu)) \preceq \epsilon^{\alpha(s) - \alpha_0(s)u}.$$

By integrating this over all such z , we get

$$\mathbf{E} [\text{Area}(\underline{A}_\epsilon^{s;u}) \mathbf{1}_{\mathcal{G}(f_t, \mu)}] \preceq \epsilon^{\alpha(s) + 1 - (\alpha_0(s) + 1)u}. \quad (4.6)$$

Similarly, suppose t, d, μ , and ϵ_0 are chosen so that (3.28) in Theorem 3.11 holds. Then for $\epsilon \in (0, \epsilon_0]$, we have

$$\mathbf{E} [\text{Area}(\underline{A}_\epsilon^{s;u}) \mathbf{1}_{\mathcal{G}(f_t, \mu)}] \succeq \epsilon^{\alpha(s) + 1 + (\alpha_0(s) + 1)u}. \quad (4.7)$$

By the change of variables formula we have

$$\text{Area}(A_\epsilon^{s;u}(\eta; t, \delta, c) \cap B_d(0)) = \int_{f_t(A_\epsilon^{s;u}(\eta; t, \delta, c) \cap B_d(0))} |(f_t^{-1})'(z)|^2 dz. \quad (4.8)$$

The Koebe quarter theorem implies

$$\underline{A}_\epsilon^{s;u/2}(\eta; t, \delta, c', d) \subset f_t(A_\epsilon^{s;u}(\eta; t, \delta, c) \cap B_d(0)) \subset \underline{A}_\epsilon^{s;2u}(\eta; t, \delta, c'', d)$$

for appropriate $c', c'' > 0$, depending only on c . Thus (4.6) implies (4.4). Similarly (4.7) implies (4.5). \square

In the remainder of this subsection we prove a result which allows us to transfer estimates between stopping times and deterministic times. We first need the following lemma.

Lemma 4.4. *Let η be a chordal SLE_κ process from $-i$ to i in \mathbf{D} . Let (f_t) be its centered Loewner maps. For $w \in \mathbf{D}$, $\delta' > 0$, $C > 1$, and $\mu \in \mathcal{M}$ let $H(w; t) = H(w; t, \delta', C, \mu)$ be the event that the following is true.*

1. $\text{dist}(w, \eta^t) \geq C^{-1}$.
2. $C^{-1} \leq |f_t'(w)| \leq C$.
3. $|f_t(w) - i|$ and $|f_t(w) + i|$ are each at least δ' .
4. The event $\mathcal{G}(f_t, \mu)$ occurs.

For $\zeta > 0$, let S_δ^ζ be the set of $w \in \mathbf{D}$ with $|w - i|, |w + i| \geq \delta$ and $1 - |w| \leq \zeta$. If $T > 0$ and $\delta > 0$, then there exists $p, \zeta, \delta' > 0$, $\mu \in \mathcal{M}$, and $C > 1$ depending on T and δ such that

$$\mathbf{P} \left(H(w; t) \quad \forall t \leq T, \quad \forall w \in S_\delta^\zeta \right) \geq p. \quad (4.9)$$

Proof. Fix $\delta' > 0$, to be determined later, and let

$$U := \{z \in \mathbf{D} : |\operatorname{Re} z| \leq \delta'/2\}.$$

Let $E_0 = E_0(T)$ be the event that $\eta^T \subset U$. By [MW14, Lemma 2.3], $\mathbf{P}(E_0) > 0$. Thus, if ζ is chosen sufficiently small and C is chosen sufficiently large, depending on δ and δ' , then condition 1 holds a.s. on E_0 for each $w \in S_{\delta'}^\zeta \supset S_\delta^\zeta$.

On E_0 , we have by Schwarz reflection that f_t a.s. extends to be conformal on a neighborhood of S_δ^ζ for each sufficiently small ζ and each $t \leq T$. It follows that for any such ζ we can find a (possibly larger) constant C depending only on δ , T , and ζ such that the probability of the event E_1 that E_0 occurs and condition 2 holds for each $w \in S_\delta^\zeta$ and each $t \in [0, T]$ is at least $\mathbf{P}(E_0)/2$.

By continuity, if δ' is chosen sufficiently small, depending only on δ and T , then the conditional probability given E_1 of the event

$$\left\{ f_t(w) \in S_{\delta'}^\zeta, \quad \forall t \leq T, \quad \forall w \in S_\delta^\zeta \right\} \quad (4.10)$$

tends to 1 as $\zeta \rightarrow 0$. In particular, we can find $\zeta > 0$ sufficiently small that the probability of the event E_2 that (4.10) occurs and that E_1 occurs is at least $\mathbf{P}(E_1)/2$.

Since η a.s. does not hit $\partial\mathbf{D}$ and f_t^{-1} is a.s. continuous, we can find $\mu \in \mathcal{M}$ such that $\mathbf{P}(\mathcal{G}(f_t, \mu) | E_2) \geq 1/2$. If $E_2 \cap \mathcal{G}(f_t, \mu)$ occurs, then the event in (4.9) occurs. \square

Lemma 4.5. *Let η be a chordal SLE_κ from $-i$ to i in \mathbf{D} with centered Loewner maps (f_t) . Let τ, τ' be stopping times for η and suppose there is a deterministic time $T > 0$ such that a.s. $\tau \leq \tau' \leq T$. For any $c > 0$, $\mu \in \mathcal{M}$, and $\delta > 0$, we can find $c' > 0$, $\delta' > 0$, and $\mu' \in \mathcal{M}$ such that for each $u > 0$, there is an $\epsilon_0 > 0$ such that for each $z \in \mathbf{D}$,*

$$\mathbf{P}(E_\epsilon^{s;u}(\eta, z; \tau, \delta, c) \cap \mathcal{G}(f_\tau, \mu)) \preceq \mathbf{P}(E_\epsilon^{s;u}(\eta, z; \tau', \delta', c') \cap G(f_{\tau'}, \mu')), \quad (4.11)$$

with the implicit constant uniform for z in compact subsets of \mathbf{D} and independent of ϵ .

Proof. Let (\mathcal{F}_t) be the filtration generated by η . Let $\hat{f}_t = f_{t+\tau} \circ f_\tau^{-1}$ and $\hat{\eta} = f_\tau(\eta|_{[\tau, \infty)})$. Fix T and δ and let p, ζ, δ', C , and μ' satisfy the conclusion of Lemma 4.4 (with μ' in place of μ). Define the events $H(f_\tau(z); t, \delta', C)$ as in Lemma 4.4 with $w = f_\tau(z)$ and f_t replaced by \hat{f}_t . That lemma together with the Markov property of SLE imply that if ϵ is chosen sufficiently small, uniformly in $z \in B_d(0)$, then the conditional probability of the event

$$\hat{H} := \{H(f_\tau(z); t, \delta', C, \mu') \quad \forall t \leq T\}$$

given $E_\epsilon^{s;u}(\eta, z; \tau, \delta, c) \cap G(f_\tau, \mu)$ is at least p . By inspection, for sufficiently small ϵ and any $t \leq T$, we have

$$E_\epsilon^{s;u}(\eta, z; \tau, \delta, c) \cap \mathcal{G}(f_\tau, \mu) \cap H(f_\tau(z); t, \delta', C, \mu') \subset E_\epsilon^{s;u}(\eta, z; \tau + t, \delta', c') \cap G(f_{t+\tau}, \mu' \circ \mu)$$

for some $c' > 0$ depending only on C and c . Since $\tau' - \tau \leq T$ a.s., it follows that

$$E_\epsilon^{s;u}(\eta, z; \tau, \delta, c) \cap \mathcal{G}(f_\tau, \mu) \cap \hat{H} \subset E_\epsilon^{s;u}(\eta, z; \tau', \delta', c') \cap G(f_{\tau'}, \mu' \circ \mu)$$

for some $c' > 0$. Taking probabilities proves (4.11) (with $\mu' \circ \mu$ in place of μ'). \square

4.3 Comparison lemmas

In this subsection we prove several lemmas comparing probabilities of sets associated with the finite time Loewner maps to probabilities of sets associated with the infinite time Loewner maps of Theorem 4.1, and use these results to estimate the areas of the sets $\mathcal{A}_\epsilon^{s;u}(\eta; c)$ of Theorem 4.1.

Our first lemma is needed for the proof of the upper bound in Theorem 4.1.

Lemma 4.6. *Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. Fix $d \in (0, 1)$. There is a $\delta > 0$ such that for each $\mu \in \mathcal{M}$ and $c > 0$, there exists $\mu' \in \mathcal{M}$ and $c' > 0$ such that for each $u > 0$, there exists $\epsilon_0 > 0$ and a bounded stopping time τ for η such that for each $z \in B_d(0)$ and each $\epsilon \in (0, \epsilon_0]$,*

$$\mathbf{P}(\mathcal{E}^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) \preceq \mathbf{P}(E_\epsilon^{s;u}(z; \tau, \delta, c') \cap \mathcal{G}(f_\tau, \mu')) \quad (4.12)$$

with the implicit constants independent of ϵ and uniform for $z \in B_d(0)$.

Proof. Suppose $\mathcal{E}^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)$ occurs. We will prove the lemma by growing some more of the curve out from $-i$ and i to get a new curve $\tilde{\eta} \stackrel{d}{=} \eta$ with the property that $E_\epsilon^{s;u}(\tilde{\eta}, z; \tau, \delta, c') \cap \mathcal{G}(f_\tau, \mu')$ occurs for an appropriate bounded stopping time τ and the derivatives of the conformal maps associated with $\tilde{\eta}^\tau$ and with η at z are comparable.

To this end, let η_0 be a chordal SLE_κ from $-i$ to i in \mathbf{D} , independent of η . Let $\bar{\eta}_0$ be its time reversal. Then $\bar{\eta}_0$ has the law of a chordal SLE_κ from i to $-i$ [Zha08b]. Let $\delta_0, C, \beta, \zeta, r, a > 0$, and $\mu_0 \in \mathcal{M}$. We assume that $\zeta \ll 1 - d$. Let P be the event that the following is true.

1. Let \bar{T} be the first time $\bar{\eta}_0$ gets within distance $e^{-\beta}$ of z . Then $\bar{T} < \infty$ and $\bar{\eta}_0^{\bar{T}}$ is disjoint from $(\mathbf{D} \setminus \mathbf{H}) \cup B_{1/2}(1)$.
2. For each $t \geq 0$, let $\phi_t : \mathbf{D} \setminus (\eta_0^t \cup \bar{\eta}_0^{\bar{T}})$ be the unique conformal map fixing z and taking $\bar{\eta}_0(\bar{T})$ to i . Let T be the first time t that $\phi_t(\eta_0(t)) = -i$ and $|\eta_0(t) - z| \leq 2e^{-\beta}$. Then $T < \infty$ and η_0^T is disjoint from $(\mathbf{D} \cap \mathbf{H}) \cup B_{1/2}(1)$.
3. Henceforth put $\phi = \phi_T$. We have $C^{-1} \leq |(\phi^{-1})'(w)| \leq C$ for each $w \in B_{(1+d)/2}(0)$.
4. We have $\phi^{-1}(B_{\delta_0}(-i) \cup B_{\delta_0}(i) \cup B_{1-r}(0)) \subset B_{(1-d)/2}(z)$.
5. Let $\bar{\sigma}$ be the last exit time of $\bar{\eta}_0$ from $B_\zeta(i)$ before time \bar{T} . Then $\bar{\eta}_0^{\bar{\sigma}} \subset B_{2\zeta}(i)$.
6. Let

$$K := \eta_0^T \cup \bar{\eta}_0([\bar{\sigma}, \bar{T}]) \cup B_{(1-d)/2}(z). \quad (4.13)$$

The harmonic measure from i of each side of $K \cap B_{(1-d)/2}(i)$ and each side of $K \cap B_{(1-d)/2}(-i)$ in the Schwarz reflection of $\mathbf{D} \setminus K$ across $[-1, 1]_{\partial \mathbf{D}}$ is at least a .

7. $\mathcal{G}'(K, \mu_0)$ occurs (Definition 2.6).

See Figure 4.2 for an illustration of the event P . In what follows, all implicit constants are required to depend only on μ, d , and the parameters for P .

First we will argue that for any choice of the parameters d, ζ , and r , we can choose the other parameters for P in such a way that $\mathbf{P}(P) \geq 1$. It follows from [MW14, Lemma 2.3] and reversibility of SLE that conditions 1, 2, and 5 hold with positive probability depending only on β, ζ , and d . By the Koebe growth theorem, if β is chosen sufficiently large (depending on r and d) and δ_0 is chosen sufficiently small (depending only on d) then condition 4 also holds simultaneously with positive probability depending only on $\beta, \zeta, d, \delta_0$, and r . By choosing C sufficiently large and a and μ_0 sufficiently small (see Lemma 2.7), depending only on d and the other parameters for P , we can arrange that the remaining conditions in the definition of P hold with probability arbitrarily close to 1. Thus we have $\mathbf{P}(P) \geq 1$.

Let $\tilde{\eta} = \eta_0$ on the event that P does not occur. On P , let $\tilde{\eta} = \phi^{-1}(\eta) \cup \eta_0^T \cup \bar{\eta}_0^{\bar{T}}$, parametrized in such a way that its image under the conformal map from \mathbf{D} to \mathbf{H} taking $-i$ to 0, i to ∞ , and 0 to i is parametrized by capacity. By the Markov property and reversibility of SLE, $\tilde{\eta}$ has the same law as η . Let (f_t) be the centered Loewner maps for $\tilde{\eta}$. Let

$$\tilde{\mathcal{E}} = \mathcal{E}_\epsilon^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu) \cap P.$$

Let τ be the hitting time of $B_\zeta(i)$ by $\tilde{\eta}$. Then τ is a bounded stopping time for $\tilde{\eta}$. Furthermore, if we choose ζ sufficiently small relative to d (independently of ϵ) then on the event $\tilde{\mathcal{E}}$ we have $\tilde{\eta} \setminus \tilde{\eta}^\tau = \bar{\eta}_0^{\bar{\sigma}}$, with $\bar{\sigma}$ as in condition 5 in the definition of P .

We claim that if the parameters for P are chosen appropriately (independently of ϵ and $z \in B_d(0)$) then for sufficiently small ϵ we have

$$\tilde{\mathcal{E}} \subset E_\epsilon^{s;u}(\tilde{\eta}, z; \tau, \delta, \tilde{c}) \cap \mathcal{G}(\tilde{f}_\tau, \tilde{\mu}) \quad (4.14)$$

for some $\tilde{\mu} \in \mathcal{M}$ depending only on d and some $\tilde{c} > 0$ and $\tilde{\mu} \in \mathcal{M}$, depending only on d, μ, c , and the parameters for P . Given the claim (4.14), our desired result (4.12) follows by taking probabilities and noting that P is independent from η .

By condition 4 in the definition of P , on the event $\tilde{\mathcal{E}}$ we have $\tilde{\eta}^\tau \subset K$, as in (4.13), provided r is chosen sufficiently small, depending only on μ and δ_0 . By condition 7 in the definition of P and Lemma 2.8, we

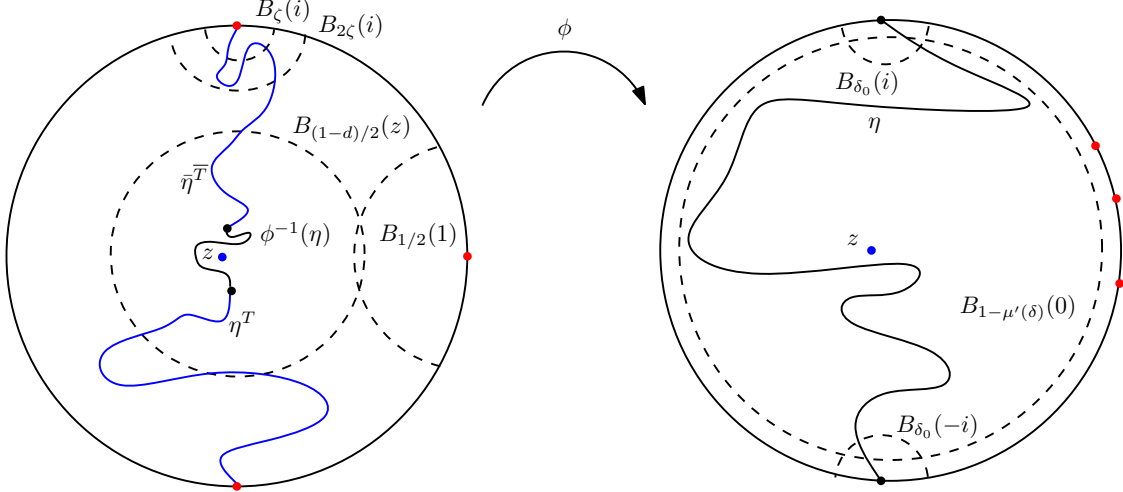


Figure 4.2: An illustration of the event P and the curve $\tilde{\eta}$ used in the proof of Lemma 4.6. The red points are $-i$, i , and 1 and their images under ϕ .

can find $\tilde{\mu} \in \mathcal{M}$ depending only on μ , d and the parameters for P such that $\tilde{\mathcal{E}} \subset \mathcal{G}(\tilde{f}_\tau, \tilde{\mu})$. By condition 6 in the definition of P , we can find $\delta > 0$ depending only on a such that $\tilde{f}_\tau(z)/|\tilde{f}_\tau(z)|$ lies at distance at least δ from $\pm i$ on $\tilde{\mathcal{E}}$. That is, condition 3 in the definition of $E_\epsilon^{s;u}(\tilde{\eta}, z; \tau, \delta, \tilde{c})$ holds on $\tilde{\mathcal{E}}$.

By condition 3 in the definition of P , we have $\text{dist}(z, \tilde{\eta}) \asymp \text{dist}(z, \eta)$ on P . It therefore follows that condition 1 in the definition of $E_\epsilon^{s;u}(\tilde{\eta}, z; \tau, \delta, \tilde{c})$ holds on $\tilde{\mathcal{E}}$ for some $\tilde{c} \asymp 1$.

It remains to show that condition 1 in the definition of $E_\epsilon^{s;u}(\tilde{\eta}, z; \tau, \delta, \tilde{c})$ holds on $\tilde{\mathcal{E}}$ provided $\tilde{c} \asymp 1$ is chosen sufficiently large. It is enough to show $|\tilde{f}'_\tau(z)| \asymp |\Psi'_\eta(z)|$ on $\tilde{\mathcal{E}}$. We will do this in two stages. Let $\Psi_{\tilde{\eta}}$ be as in Section 4.1 with $\tilde{\eta}$ in place of η . First we will show that $|\Psi'_\eta(z)| \asymp |\Psi'_{\tilde{\eta}}(z)|$, and then we will show that $|\Psi'_{\tilde{\eta}}(z)| \asymp |\tilde{f}'_\tau(z)|$.

For the first stage, let g be the conformal automorphism of \mathbf{D} taking $\Psi_\eta(\phi(-i^+))$ to $-i$, $\Psi_\eta(\phi(i^-))$ to i , and $\Psi_\eta(\phi(1))$ to 1 . Then we have

$$\Psi_{\tilde{\eta}} = g \circ \Psi_\eta \circ \phi. \quad (4.15)$$

By condition 7 in the definition of P , together with the definition of $\tilde{\mathcal{E}}$, we have $|g'| \asymp 1$ uniformly on \mathbf{D} on $\tilde{\mathcal{E}}$, so by condition 3 in the definition of P , we have $|\Psi'_{\tilde{\eta}}(z)| \asymp |\Psi'_\eta(z)|$ on $\tilde{\mathcal{E}}$.

For the second stage, let $\Psi_{\tilde{\eta}^\tau}$ be the conformal map from $\mathbf{D} \setminus \tilde{\eta}^\tau$ to \mathbf{D} taking $-i^+$ to $-i$ and fixing i and 1 . Then $\Psi_{\tilde{\eta}^\tau}$ differs from \tilde{f}_τ by a conformal automorphism of \mathbf{D} taking $\tilde{f}_\tau(-i^+)$ to $-i$ and $\tilde{f}_\tau(1)$ to 1 . Since $\mathcal{G}(\tilde{f}_\tau, \tilde{\mu})$ holds on $\tilde{\mathcal{E}}$, we have

$$|\Psi'_{\tilde{\eta}^\tau}(z)| \asymp |\tilde{f}'_\tau(z)|. \quad (4.16)$$

Let I be the arc of $\partial\mathbf{D}$ of length ζ centered at 1 . By condition 7 in the definition of P (c.f. Remark B.2), the lengths of $\Psi_{\tilde{\eta}}(I)$ and $\Psi_{\tilde{\eta}^\tau}(I)$ are ≥ 1 on $\tilde{\mathcal{E}}$. By conditions 1, 4, and 5 in the definition of P and a study of the harmonic measure from 1 in the Schwarz reflection of $D_{\tilde{\eta}}$, the distances from $\Psi_{\tilde{\eta}}(z)$ to $\Psi_{\tilde{\eta}}(I)$ and from $\Psi_{\tilde{\eta}^\tau}(z)$ to $\Psi_{\tilde{\eta}^\tau}(I)$ are ≥ 1 on $\tilde{\mathcal{E}}$ provided ζ is chosen sufficiently small relative to d . By Lemma B.1, it holds on $\tilde{\mathcal{E}}$ that

$$|\Psi'_{\tilde{\eta}}(z)| \asymp \frac{\text{hm}^z(I; D_{\tilde{\eta}})}{\text{dist}(z, \tilde{\eta})} \quad \text{and} \quad |\Psi'_{\tilde{\eta}^\tau}(z)| \asymp \frac{\text{hm}^z(I; \mathbf{D} \setminus \tilde{\eta}^\tau)}{\text{dist}(z, \tilde{\eta}^\tau)}. \quad (4.17)$$

By conformal invariance $\text{hm}^z(I; D_{\tilde{\eta}})$ is the same as the probability that a Brownian motion started from $\Psi_{\tilde{\eta}^\tau}(z)$ exits \mathbf{D} in $\Psi_{\tilde{\eta}^\tau}(I)$ before hitting $\Psi_{\tilde{\eta}^\tau}(\tilde{\eta}([\tau, \infty))$. By conditions 5 and 6 in the definition of P , if ζ is chosen sufficiently small, independently of ϵ , then on $\tilde{\mathcal{E}}$, the distance from $\Psi_{\tilde{\eta}^\tau}(\tilde{\eta}([\tau, \infty))$ to $\Psi_{\tilde{\eta}^\tau}(z) \cup \Psi_{\tilde{\eta}^\tau}(I)$ is at least a deterministic ϵ -independent constant; and the diameter of $\Psi_{\tilde{\eta}^\tau}(\tilde{\eta}([\tau, \infty))$ is smaller than $1/100$ times this constant (here we again use harmonic measure from 1). Therefore, the probability that a Brownian

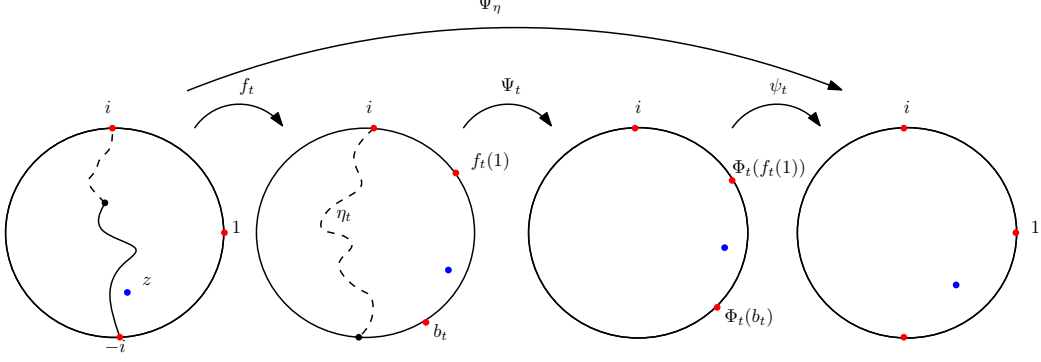


Figure 4.3: An illustration of the maps used in the proof of Lemma 4.7 for the right side of **D**. The red points are the images of $-i, i$, and 1 under the various maps. The last map ψ_t takes these points back to their original positions so that by composing all three maps we recover the original map Ψ_η .

motion started from $\Psi_{\tilde{\eta}^\tau}(z)$ exits **D** in $\Psi_{\tilde{\eta}^\tau}(I)$ before hitting $\Psi_{\tilde{\eta}^\tau}(\tilde{\eta}([\tau, \infty))$ is proportional to the probability that a Brownian motion started from $\Psi_{\tilde{\eta}^\tau}(z)$ exits **D** in $\Psi_{\tilde{\eta}^\tau}(I)$. That is, $\text{hm}^z(I; D_{\tilde{\eta}}) \asymp \text{hm}^z(I; \mathbf{D} \setminus \tilde{\eta}^\tau)$ on $\tilde{\mathcal{E}}$. By combining this with (4.16) and (4.17), we conclude. \square

The next lemma is needed for the proof of the lower bound in Theorem 4.1.

Lemma 4.7. *Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. Fix $d \in (0, 1)$. For each $\delta > 0$, $\mu \in \mathcal{M}$, and $c > 0$, there exists $\mu' \in \mathcal{M}$ and $c' > 0$ such that for each $u > 0$, there exists $\epsilon_0 = \epsilon_0(c, c', u, \delta, \mu, \mu', d) > 0$ such that for $z \in B_d(0)$ and $\epsilon \in (0, \epsilon_0]$,*

$$\mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c') \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')) \succeq \mathbf{P}(E_\epsilon^{s;u}(\eta, z; t, \delta, c) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu)), \quad (4.18)$$

with implicit constants independent of ϵ and uniform for $z \in B_d(0)$.

Proof. Let (f_t) be the centered Loewner maps for η as in Section 4.2. For $t \geq 0$, let $\eta_t = f_t(\eta|_{[t, \infty)})$. Let D_t be the connected component of $\mathbf{D} \setminus \eta_t$ containing 1 on its boundary and let D_t^- be the other connected component of $\mathbf{D} \setminus \eta_t$. Let $\Psi_t : D_t \rightarrow \mathbf{D}$ (resp. $\Psi_t^- : D_t^- \rightarrow \mathbf{D}$) be the unique conformal maps fixing $-i, i, 1$ (resp. $-i, i, -1$). Let b_t (resp. b_t^-) be the image of the right (resp. left) side of $-i$ under f_t . Finally, let ψ_t (resp. ψ_t^-) be the conformal automorphism of \mathbf{D} fixing i , taking $\Psi_t(b_t)$ to $-i$, and taking $\Psi_t(f_t(1))$ to 1 (resp. fixing i , taking $\Psi_t^-(b_t^-)$ to $-i$, and taking $\Psi_t^-(f_t(-1))$ to -1). Then for each t ,

$$\Psi_\eta = \psi_t \circ \Psi_t \circ f_t, \quad \Psi_\eta^- = \psi_t^- \circ \Psi_t^- \circ f_t. \quad (4.19)$$

Moreover, (Ψ_t, Ψ_t^-) and f_t are independent and we have $\Psi_t \stackrel{d}{=} \Psi_\eta$, $\Psi_t^- \stackrel{d}{=} \Psi_\eta^-$. See Figure 4.3 for an illustration of some of these maps.

For $C > 1$, $\mu' \in \mathcal{M}$, and $w \in \mathbf{D}$, let $F(w) = F(w; t, C, \mu')$ be the event that $w \in D_t$, $C^{-1} \leq |\Psi_t'(w)| \leq C$, $\text{dist}(w, \eta_t) = \text{dist}(w, \partial \mathbf{D})$, and $\mathcal{G}(\Psi_t, \mu') \cap \mathcal{G}(\Psi_t^-, \mu')$ occurs. By [MW14, Lemma 2.3], for each $\delta > 0$, we can find $C > 1$ and $\mu' \in \mathcal{M}$ such that for each $w \in \mathbf{D}$ lying at distance at least δ from $\pm i$ with $\text{Re } w \geq 0$, we have that $\mathbf{P}(F(w)) \succeq 1$, with the implicit constant independent of ϵ and uniform for w satisfying the conditions above.

If we let

$$F^*(z) := E_\epsilon^{s;u}(\eta, z; t, \delta, c) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu) \cap F(f_t(z)),$$

then by independence of f_t and η_t and our choice of parameters for $F(\cdot)$ we have

$$\mathbf{P}(F^*(z)) \asymp \mathbf{P}(E_\epsilon^{s;u}(\eta, z; t, \delta, c) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu)). \quad (4.20)$$

By the “ \mathcal{G} ” condition in the definition of $F(f_t(z))$, we have that $|\psi_t'|$ and $|(\psi_t^-)'|$ are bounded above and below by positive ϵ -independent constants on the event $F^*(z)$. Hence it follows from (4.19) that $F^*(z) \subset \mathcal{E}_\epsilon^{s;u}(\eta, z; c') \cap \mathcal{G}(\Psi_\eta, \mu'') \cap \mathcal{G}(\Psi_\eta^-, \mu'')$ for some $c' > 0$ and some $\mu'' \in \mathcal{M}$ which do not depend on ϵ and are uniform for $z \in B_d(0)$. By combining this with (4.20) we get (4.18) (with μ'' in place of μ'). \square

Now we can transfer our area estimates for the finite time sets to area estimates for the time infinity sets.

Lemma 4.8. *Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. For each $d \in (0, 1)$, each $\mu \in \mathcal{M}$, and each $c > 0$,*

$$\mathbf{E} \left(\text{Area}(\mathcal{A}_\epsilon^{s;u}(\eta; c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)} \right) \preceq \epsilon^{\gamma(s) - \gamma_0(s)u}. \quad (4.21)$$

Furthermore, there exists $d \in (0, 1)$ such that for each $c > 0$, there exists $\mu \in \mathcal{M}$ and $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0]$,

$$\mathbf{E} \left(\text{Area}(\mathcal{A}_\epsilon^{s;u}(\eta; c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)} \right) \succeq \epsilon^{\gamma(s) + \frac{3\gamma_0(s)}{2}u}. \quad (4.22)$$

In both (4.21) and (4.22) the implicit constants depend on the other parameters but not on ϵ .

Proof. The relation (4.21) follows by integrating the estimate from Lemma 4.6 over $B_d(0)$, applying Lemma 4.5 to replace the stopping time τ with a deterministic time, then applying (4.4) from Lemma 4.3.

For (4.22), first choose t, μ, c_0 , and d in such a way that (4.5) from Lemma 4.3) holds. By Lemma 4.7, we can find $c'_0 > 0$, $\mu' \in \mathcal{M}$, and $\tilde{\epsilon}_0 > 0$ such that for $\epsilon \in (0, \tilde{\epsilon}_0]$ and $z \in B_d(0)$, we have

$$\mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c'_0) \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')) \geq \mathbf{P}(E_\epsilon^{s;u}(\eta, z; t, \delta, c_0) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu)).$$

We then integrate this estimate over $B_d(0)$ to get

$$\begin{aligned} & \mathbf{E} \left(\text{Area}(\mathcal{A}_\epsilon^{s;u}(\eta; c'_0) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')} \right) \\ & \geq \mathbf{E} \left(\text{Area}(\{z \in A_\epsilon^{s;u}(t, \delta, c_0) \cap B_d(0) : \text{Re } f_t(z) \geq 0\}) \mathbf{1}_{\mathcal{G}(f_t, \mu)} \right). \end{aligned} \quad (4.23)$$

Since the law of (f_t) is symmetric about the imaginary axis the right side of (4.23) is at least

$$\frac{1}{2} \mathbf{E} \left(\text{Area}(\hat{A}_\epsilon^{s;u}(t, \delta, c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(f_t, \mu)} \right).$$

By combining this with (4.5) from Lemma 4.3, replacing u with $\frac{2}{3}u$, and choosing $\epsilon_0 \in (0, \tilde{\epsilon}_0]$ such that $\epsilon_0^{u/3} \leq c/c_0$, we obtain (4.22). \square

4.4 Proof of Theorem 4.1

To deduce Theorem 4.1 from the area estimate of Lemma 4.8, we need to argue that the probabilities of the events of Theorem 4.1 do not depend too strongly on z . This is accomplished in the next lemma.

Lemma 4.9. *Suppose we are in the setting of Theorem 4.1 with $x = -i$, $y = i$. Fix $d \in (0, 1)$. For any $\mu \in \mathcal{M}$ and $c > 0$, we can find $\mu' \in \mathcal{M}$ and $c' > 0$ such that for each $z, w \in B_d(0)$ and $\epsilon \in (0, 1)$, we have*

$$\mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, w; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) \preceq \mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c') \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')) \quad (4.24)$$

with implicit constants independent of ϵ and uniform in $B_d(0)$.

Proof. The basic idea of the proof is as follows. First we apply a conformal map taking z to w and fixing $-i$. The image of η under such a map will be an SLE_κ with a new target point b . To compare such a curve to our original curve, we grow a carefully chosen segment of the new curve backward from b in such a way that when we map back to \mathbf{D} , we get a chordal SLE_κ from $-i$ to i . We now commence with the details.

For $z, w \in B_d(0)$, let $\phi = \phi_{z,w} : \mathbf{D} \rightarrow \mathbf{D}$ be the unique conformal map fixing $-i$ and taking z to w . Let $b := \phi(i)$ and $\eta^b = \phi(\eta)$. The law of η^b is that of a chordal SLE_κ process from $-i$ to b in \mathbf{D} .

The map ϕ depends continuously on z in the topology of uniform convergence on compacts. It follows that for any $\mu \in \mathcal{M}$ we can find a deterministic constant $c' > 0$ depending only on c, μ , and d , linearly on c , and a deterministic $\mu' \in \mathcal{M}$ depending only on μ and d such that for $z, w \in B_d(0)$,

$$\mathcal{E}_\epsilon^{s;u}(\eta^b, w; c) \cap \mathcal{G}(\Psi_{\eta^b}, \mu) \cap \mathcal{G}(\Psi_{\eta^b}^-, \mu) \subset \mathcal{E}_\epsilon^{s;u}(\eta, z; c') \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu'). \quad (4.25)$$

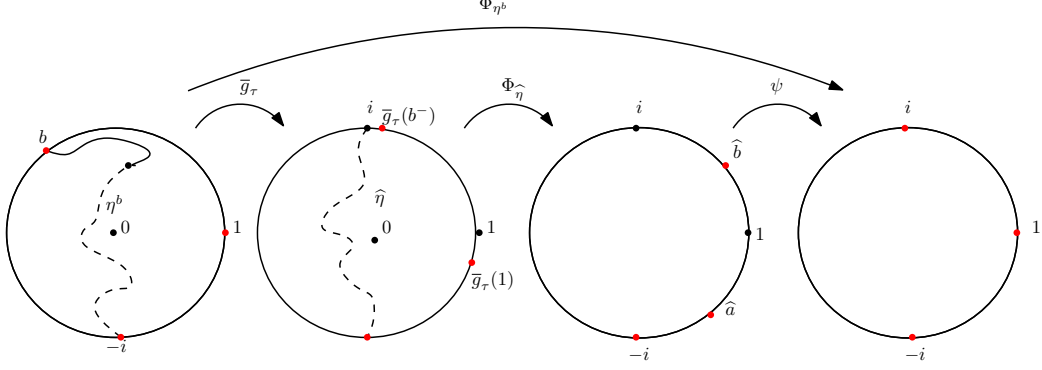


Figure 4.4: An illustration of the maps used in the proof of Lemma 4.9 on the event \overline{E}^b .

Let $\bar{\eta}^b$ be the time reversal of η^b . Then $\bar{\eta}^b$ is a chordal SLE_κ from b to $-i$ in \mathbf{D} [Zha08b]. We give $\bar{\eta}^b$ the usual chordal parametrization, so that it is the conformal image of a chordal SLE_κ parametrized by capacity from 0 to ∞ in \mathbf{H} . For each $t \geq 0$, let $\bar{g}_t : \mathbf{D} \setminus \bar{\eta}^b([0, t]) \rightarrow \mathbf{D}$ be the unique conformal map fixing $-i$ and w . Let τ be the first time t that $\bar{g}_t(\bar{\eta}^b(t)) = i$.

Fix $\mu^b \in \mathcal{M}$ and let \overline{E}^b be the event that τ is less than or equal to the first time t that $\bar{\eta}^b$ hits $B_{d^*}(0)$, where

$$d^* := 1 - \frac{1}{4} \inf_{z, w \in B_d(0)} \text{dist}(\phi_{z, w}(B_d(0)), \partial \mathbf{D});$$

and the event $\mathcal{G}(\bar{g}_\tau, \mu^b)$ occurs. By [MW14, Lemma 2.3], if μ^b is chosen sufficiently small then $\mathbf{P}(\overline{E}^b)$ is a positive constant depending only on μ^b and $B_d(0)$.

By the Markov property, conditional on \overline{E}^b , the law of $\bar{g}_\tau(\bar{\eta}^b|_{[\tau, \infty)})$ is that of a chordal SLE_κ process from i to $-i$ in \mathbf{D} . Therefore its time reversal $\hat{\eta} := \bar{g}_\tau^{-1}(\bar{\eta}^b|_{[0, \tau]})$, where τ^b is the time corresponding to τ under the time reversal, has the law of a chordal SLE_κ from $-i$ to i in \mathbf{D} . In particular, $\hat{\eta} \stackrel{d}{=} \eta$.

Define the open sets $D_{\eta^b}, D_{\hat{\eta}}$ and the maps $\Psi_{\eta^b}, \Psi_{\hat{\eta}}$ as in Section 4.1 with $\eta^b, \hat{\eta}$, resp., in place of η . Let ψ (resp. ψ^-) be the conformal automorphism of \mathbf{D} which fixes $-i$, takes $(\Psi_{\hat{\eta}} \circ \bar{g}_\tau)(i)$ to i , and takes $(\Psi_{\hat{\eta}} \circ \bar{g}_\tau)(1)$ to 1 (resp. fixes $-i$, takes $(\Psi_{\hat{\eta}}^- \circ \bar{g}_\tau)(b^+)$ to i , and takes $(\Psi_{\hat{\eta}}^- \circ \bar{g}_\tau)(-1)$ to -1). Then we have

$$\Psi_{\eta^b} = \psi \circ \Phi_{\hat{\eta}} \circ \bar{g}_\tau, \quad \Psi_{\eta^b}^- = \psi^- \circ \Psi_{\hat{\eta}}^- \circ \bar{g}_\tau.$$

See Figure 4.4 for an illustration of some of these maps.

Since $\mathcal{G}(\bar{g}_\tau, \mu^b) \subset \overline{E}^b$, on the event $\overline{E}^b \cap \mathcal{E}_\epsilon^{s;u}(\hat{\eta}, w; c) \cap \mathcal{G}(\Psi_{\hat{\eta}}, \mu) \cap \mathcal{G}(\Psi_{\hat{\eta}}^-, \mu)$, it holds that $|\psi'|$ and $|(\psi^-)'|$ are bounded above and below by deterministic positive constants depending only on μ^b and μ . Furthermore, we have that $\mathcal{G}(\psi, \mu_2) \cap \mathcal{G}(\psi^-, \mu_2)$ holds for some $\mu_2 \in \mathcal{M}$ depending on μ^b, μ . The Koebe distortion theorem and the definition of \overline{E}^b imply that $|g'_\tau(w)|$ is bounded above and below by positive constants depending only on d on the event \overline{E}^b . Hence for some $c' > 0$, independent of ϵ and uniform for $z, w \in B_d(0)$, we have

$$\overline{E}^b \cap \mathcal{E}_\epsilon^{s;u}(\hat{\eta}, w; c) \cap \mathcal{G}(\Psi_{\hat{\eta}}, \mu) \cap \mathcal{G}(\Psi_{\hat{\eta}}^-, \mu) \subset \mathcal{E}_\epsilon^{s;u}(\eta^b, w; c') \cap \mathcal{G}(\Psi_{\eta^b}, \mu_2 \circ \mu \circ \mu^b) \cap \mathcal{G}(\Psi_{\eta^b}^-, \mu_2 \circ \mu \circ \mu^b). \quad (4.26)$$

By the Markov property and the fact that $\mathbf{P}(\overline{E}^b)$ is uniformly positive, we have

$$\mathbf{P}(\overline{E}^b \cap \mathcal{E}_\epsilon^{s;u}(\hat{\eta}, w; c) \cap \mathcal{G}(\Psi_{\hat{\eta}}, \mu) \cap \mathcal{G}(\Psi_{\hat{\eta}}^-, \mu)) \asymp \mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta^b, w; c, \lambda, \ell) \cap \mathcal{G}(\Psi_{\eta^b}, \mu) \cap \mathcal{G}(\Psi_{\eta^b}^-, \mu)). \quad (4.27)$$

Since $\hat{\eta} \stackrel{d}{=} \eta$, (4.24) now follows from (4.26), (4.27), and (4.25). \square

Proof of Theorem 4.1. By applying a coordinate change it is enough to consider the case $x = -i$, $y = i$. By Lemma 4.9, for any $z \in B_d(0)$, we have, in the notation of that lemma,

$$\begin{aligned} \mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) &\preceq \mathbf{E} \left(\text{Area}(\mathcal{A}_\epsilon^{s;u}(\eta, z; c') \cap B_d(0)) \mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')} \right) \\ \mathbf{P}(\mathcal{E}_\epsilon^{s;u}(\eta, z; c') \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')) &\succeq \mathbf{E} \left(\text{Area}(\mathcal{A}_\epsilon^{s;u}(\eta, z; c) \cap B_d(0)) \mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)} \right). \end{aligned}$$

We conclude by combining this with Lemma 4.8 (and slightly decreasing u and shrinking ϵ_0 as in the proof of Lemma 4.8 to get a small enough constant in the event for lower bound). \square

4.5 Finite time estimates

In this subsection we use Theorem 4.1 and the comparison lemmas of Section 4.3 to prove estimates for the finite time Loewner maps. The result of this subsection is not needed for the proof of our main result, and is stated only for the sake of completeness.

Theorem 4.10. *Let $\kappa \in (0, 4]$. Let (f_t) be the centered Loewner maps of a chordal SLE_κ process η from $-i$ to i in \mathbf{D} . Fix $d \in (0, 1)$. Define the events $E_\epsilon^{s;u}(z; t, \delta, c)$ as in Definition 4.2 and the sets $G(f_t, \mu)$ as in Definition 2.5. For any $\mu \in \mathcal{M}$, $t, \delta, c > 0$, $\epsilon > 0$, and $z \in B_d(0)$,*

$$\mathbf{P}(E_\epsilon^{s;u}(\eta, z; t, \delta, c) \cap \mathcal{G}(f_t, \mu) \cap \{\text{Re } f_t(z) \geq 0\}) \preceq \epsilon^{\gamma(s) - 2\gamma_0(s)u}. \quad (4.28)$$

Moreover, there exists $t_* > 0$, $\delta > 0$, and $\mu \in \mathcal{M}$ such that for each $c > 0$ and each $u > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$ and $z \in B_d(0)$,

$$\mathbf{P}(E_\epsilon^{s;u}(\eta, z; t, \delta, c) \cap \mathcal{G}(f_t, \mu)) \succeq \epsilon^{\gamma(s) + 2\gamma_0(s)u}. \quad (4.29)$$

In (4.28) and (4.29) the implicit constants are independent of ϵ and uniform for $z \in B_d(0)$. The estimate (4.28) holds with t replaced by a bounded stopping time. The estimate (4.29) holds with t replaced by a bounded stopping time which is a.s. $\geq t_*$.

Proof. The statement for deterministic times follows by combining Theorem 4.1 with Lemmas 4.5, 4.6 and 4.7. The statement for stopping times follows from this and Lemma 4.5. \square

5 Upper bounds for multifractal and integral means spectra

In this section we will use the upper bounds in Theorems 3.1 and 4.1 to prove the Hausdorff dimension upper bounds in Theorem 1.1 as well the upper bound in Corollary 1.8.

5.1 Upper bound for the Hausdorff dimension of the subset of the circle

In this subsection we use Proposition 3.1 to obtain upper bounds on the Hausdorff dimension of the sets $\tilde{\Theta}^s(\mathbf{D} \setminus K_t)$ of Section 1.1 for the hulls (K_t) of a chordal SLE_κ from $-i$ to i in \mathbf{D} . In light of Lemma 2.16, Proposition 5.1 implies the upper bounds for $\dim_{\mathcal{H}} \tilde{\Theta}^{s;\geq}(D_\eta)$ and $\dim_{\mathcal{H}} \tilde{\Theta}^{s;\leq}(D_\eta)$ in Theorem 1.1.

Proposition 5.1. *Let η be a chordal SLE_κ process from $-i$ to i in \mathbf{D} with centered Loewner maps (f_t) (defined as in Section 3.5) and hulls (K_t) . Let $\tilde{\xi}(s)$, s_- , and s_+ be as in (1.3). For each $t > 0$ and $s \in [-1, 1]$, a.s.*

$$\begin{aligned} \dim_{\mathcal{H}} \tilde{\Theta}^{s;\geq}(\mathbf{D} \setminus K_t) &\leq \tilde{\xi}(s), & 0 \leq s \leq s_+ \\ \dim_{\mathcal{H}} \tilde{\Theta}^{s;\leq}(\mathbf{D} \setminus K_t) &\leq \tilde{\xi}(s), & s_- \leq s \leq 0. \end{aligned} \quad (5.1)$$

Almost surely, for each $s \notin [s_-, s_+]$ we have $\tilde{\Theta}^s(\mathbf{D} \setminus K_t) = \emptyset$.

Remark 5.2. If $\alpha(s)$ is as in (3.2) in the statement of Theorem 3.1, we have $\tilde{\xi}(s) = 1 - \alpha(s)$.

Proof of Proposition 5.1. For $\delta > 0$ and $s \in (-1, 1)$, let

$$\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t) := \tilde{\Theta}^{s;*}(\mathbf{D} \setminus K_t) \cap \{x \in \partial\mathbf{D} : |x - i|, |x + i| \geq \delta, \quad 1 - |f_t^{-1}(x)| \geq \delta\},$$

where $*$ stands for \geq in the case $s \geq 0$ or \leq in the case $s < 0$. The reason for this definition is that it will allow us to apply the estimates of Proposition 3.6 after a change of coordinates from \mathbf{D} to \mathbf{H} . By countable stability of Hausdorff dimension, to prove (5.1), it is enough to show that a.s.

$$\mathcal{H}^\beta(\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t)) = 0 \quad \forall \delta > 0, \quad \forall \beta > \tilde{\xi}(s).$$

Henceforth fix δ , β , and s as above. Also let $s' \in [0, s)$ (if $s \geq 0$) or $s' \in (s, 0)$ (if $s < 0$) be chosen in such a way that $\tilde{\xi}(s') < \beta$.

For $n \in \mathbf{N}$ and $k \in \{1, \dots, n\}$, let

$$B_n^k := \left\{ w \in \mathbf{D} : \frac{\pi(k-1)}{2^{n-1}} \leq \arg w \leq \frac{\pi k}{2^{n-1}}, \quad 2^{-n} \leq 1 - |w| \leq 2^{-n+1} \right\}. \quad (5.2)$$

Let E_n^k be the event there is a $w \in B_n^k$ with $1 - |f_t^{-1}(w)| \geq \delta/2$ and

$$\begin{cases} |(f_t^{-1})'(w)| \geq 2^{ns'}, & s \geq 0 \\ |(f_t^{-1})'(w)| \leq 2^{ns'}, & s < 0. \end{cases} \quad (5.3)$$

Each B_n^k can be covered by at most an (n, k) -independent constant number of balls of radius $< 2^{-n-1}$, and each point of B_n^k lies at distance at least 2^{-n} from $\partial\mathbf{D}$. So the Koebe distortion and growth theorems imply that for sufficiently large n , on the event E_n^k , we have that for the center z of one of these balls, $|(f_t^{-1})'(z)|$ is at least (if $s \geq 0$) or at most (if $s < 0$) an (n, k) -independent constant times $2^{ns'}$; $|f_t^{-1}(z)| \leq \delta$; and $1 - |f_t^{-1}(z)| \geq \delta$.

For $n \in \mathbf{N}$, let \mathcal{K}_n be the set of those $k \in \{1, \dots, n\}$ such that $\exp(i\pi k/2^{n-1})$ lies at distance at least $\delta/2$ from $-i$ and i . By Proposition 3.6 and a change of coordinates to \mathbf{H} , whenever $k \in \mathcal{K}_n$, we have

$$\mathbf{P}(E_n^k) \preceq 2^{-n(1-\tilde{\xi}(s'))} \quad (5.4)$$

where the implicit constant is independent of n and uniform for $k \in \mathcal{K}_n$.

For $n \in \mathbf{N}$ and $k \in \{1, \dots, n\}$, let

$$I_n^k := \left\{ x \in \partial\mathbf{D} : \frac{\pi(k-1)}{2^{n-1}} \leq \arg x \leq \frac{\pi k}{2^{n-1}} \right\}.$$

For $m \in \mathbf{N}$, let \mathcal{I}_m be the collection of those intervals I_n^k for pairs (n, k) such that $n \geq m$, $k \in \mathcal{K}_n$, and E_n^k occurs. We claim that for each $m \in \mathbf{N}$, \mathcal{I}_m is a cover of $\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t)$. Indeed, if $x \in \tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t)$, then for any $m \in \mathbf{N}$ we can find $n \geq m$ and $w \in \mathbf{D}$ with $1 - |w| \leq 2^{-n}$, $\arg w = \arg x$, $|(f_t^{-1})'(w)| \geq (1 - |w|)^{-s'}$ (resp. $|(f_t^{-1})'(w)| \leq (1 - |w|)^{-s'}$ if $s < 0$), and $1 - |f_t^{-1}(w)| \geq \delta/2$. The point w lies in B_n^k for some pair (n, k) with $I_{n,k} \in \mathcal{I}_m$. Since $\arg w = \arg x$, we have $x \in I_{n,k}$ for this choice of (n, k) .

Now, observe that (5.4) implies

$$\begin{aligned} \mathbf{E} \left(\sum_{I \in \mathcal{I}_m} (\text{diam } I)^\beta \right) &\asymp \sum_{n=m}^{\infty} \sum_{k \in \mathcal{K}_n} 2^{-n\beta} \mathbf{P}(E_n^k) \\ &\preceq \sum_{n=m}^{\infty} 2^{-n(\beta - \tilde{\xi}(s'))}. \end{aligned} \quad (5.5)$$

This tends to 0 as $m \rightarrow \infty$ since $\beta > \tilde{\xi}(s')$ (by our choice of parameters above). Since \mathcal{I}_m is a covering of $\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t)$ by intervals of diameter tending to zero as $m \rightarrow \infty$, this proves $\mathcal{H}^\beta(\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t)) = 0$.

If $s \in [-1, 1] \setminus [s_-, s_+]$, then $\tilde{\xi}(s) < 0$, so the right side of (5.5) for $\beta = 0$ decays exponentially fast in m . Thus the expected number of sets in \mathcal{I}_m tends to zero exponentially fast, and it follows from the Borel Cantelli lemma that a.s. $\mathcal{I}_m = \emptyset$ for sufficiently large m . Hence a.s. $\tilde{\Theta}_\delta^{s;*}(\mathbf{D} \setminus K_t) = \emptyset$ for each $\delta > 0$. Finally, it follows from Lemma 2.11 that $\tilde{\Theta}^{s;*}(\mathbf{D} \setminus K_t) = \emptyset$ for $s \notin [-1, 1]$. \square

5.2 Upper bound for the Hausdorff dimension of the subset of the curve

In this subsection we will use Theorem 4.1 to give an upper bound for the Hausdorff dimension of the sets $\Theta^{s;\geq}(D)$ and $\Theta^{s;\leq}(D)$ of Section 1.1, with $D = D_\eta$ as in Theorem 1.1. For this purpose it will be convenient to introduce a slight variant of the sets of Section 1.1. For a domain $D \subset \mathbf{C}$, a conformal map $\phi : \mathbf{D} \rightarrow D$, $s \in \mathbf{R}$, and $u > 0$, let

$$\Theta^{s;u}(D) := \left\{ x \in \partial D : s - u \leq \limsup_{\epsilon \rightarrow 0} \frac{\log |\phi'((1-\epsilon)\phi^{-1}(x))|}{-\log \epsilon} \leq s + u \right\}. \quad (5.6)$$

Lemma 5.3. *Let η be a chordal SLE $_\kappa$ from $-i$ to i in \mathbf{D} and let D_η , $\xi(s)$, s_- , and s_+ be as in Theorem 1.1. Then a.s.*

$$\dim_{\mathcal{H}} \Theta^{s;u}(D_\eta) \leq \xi(s) + o_u(1), \quad (5.7)$$

whenever $s \in [s_-, s_+]$, and a.s. $\Theta^{s;u}(D_\eta) = \emptyset$ for sufficiently small u otherwise. The $o_u(1)$ in (5.7) tends to 0 as $u \rightarrow 0$ and can be taken to be uniform for s in compact subsets of $(-1, 1)$.

Remark 5.4. If $\alpha(s)$ is as in (3.2), $\gamma(s)$ is as in (4.1), and $\xi(s)$ is as in (1.4), we have

$$\xi(s) = 2 - \frac{\gamma(s)}{1-s} = \frac{1-\alpha(s)}{1-s}. \quad (5.8)$$

To prove Lemma 5.3 we first need the following lemma.

Lemma 5.5. *Let $D \subset \mathbf{C}$ be a simply connected domain and let $\phi : \mathbf{D} \rightarrow D$ be a conformal map. Suppose $x \in \Theta^{s;u}(D)$ for some $s \in \mathbf{R}$ and $u > 0$. There is a sequence of points (w_k) in D converging to x such that*

$$\frac{-s-u}{1-s+u} \leq \liminf_{k \rightarrow \infty} \frac{\log |(\phi^{-1})'(w_k)|}{-\log \text{dist}(w_k, \partial D)} \leq \limsup_{k \rightarrow \infty} \frac{\log |(\phi^{-1})'(w_k)|}{-\log \text{dist}(w_k, \partial D)} \leq \frac{-s+u}{1-s-u}.$$

and

$$\limsup_{k \rightarrow \infty} \frac{\log |w_k - x|}{-\log \text{dist}(w_k, \partial D)} \leq -\frac{1-s-u}{1-s+u}.$$

Proof. Let $x \in \Theta^{s;u}(D)$. For $\epsilon > 0$, put $z_\epsilon = \phi((1-\epsilon)\phi^{-1}(x))$. By the Koebe quarter theorem, we have

$$\text{dist}(z_\epsilon, \partial D) \asymp \epsilon |\phi'((1-\epsilon)\phi^{-1}(x))|, \quad (5.9)$$

with proportionality constants $1/4$ and 4 in each \asymp . Clearly,

$$(\phi^{-1})'(z_\epsilon) = \frac{1}{\phi'((1-\epsilon)\phi^{-1}(x))}. \quad (5.10)$$

Now let $\delta > 0$. By [JVL12, Proposition 2.7] we have

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \epsilon^{1-s-u}}{\log v(x; \epsilon)} \leq 1,$$

where $v(x; \epsilon)$ is the length of the image of the curve $t \mapsto z_t$ for $t \in [0, \epsilon]$. Consequently, for sufficiently small $\epsilon > 0$ we have

$$|z_\epsilon - x| \leq \epsilon^{1-s-u-\delta}. \quad (5.11)$$

By assumption, for sufficiently small $\epsilon > 0$ we have $|\phi'((1-\epsilon)\phi^{-1}(x))| \leq \epsilon^{-s-u-\delta}$, so by (5.9), $\text{dist}(z_\epsilon, \partial D) \leq \epsilon^{1-s-u-\delta}$ and by (5.10), $|(\phi^{-1})'(z_\epsilon)| \geq \epsilon^{s+u+\delta}$. Furthermore, for any $k \in \mathbf{N}$, we can find ϵ_k such that $|\phi'((1+\epsilon_k)\phi^{-1}(x))| \geq \epsilon_k^{-s+u+\delta}$. For each such k , we have $\text{dist}(z_{\epsilon_k}, \partial D) \geq \epsilon_k^{1-s+u+\delta}$ and $|(\phi^{-1})'(z_{\epsilon_k})| \leq \epsilon_k^{s-u-\delta}$. Hence

$$\liminf_{k \rightarrow \infty} \frac{\log |(\phi^{-1})'(z_{\epsilon_k})|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \geq \frac{-s-u-\delta}{1-s+u+\delta}. \quad (5.12)$$

and

$$\limsup_{k \rightarrow \infty} \frac{\log |(\phi^{-1})'(z_{\epsilon_k})|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \leq \frac{-s+u+\delta}{1-s-u-\delta}. \quad (5.13)$$

By (5.11), we also have

$$\limsup_{k \rightarrow \infty} \frac{\log |z_{\epsilon_k} - x|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \leq -\frac{1-s-u-\delta}{1-s+u+\delta}. \quad (5.14)$$

Since δ is arbitrary, by combining (5.12), (5.14), and (5.13) we obtain the statement of the lemma with $w_k = z_{\epsilon_k}$. \square

Proof of Lemma 5.3. The statement for $s \notin [s_-, s_+]$ follows from the analogous statement in Proposition 5.1, so we henceforth assume $s \in [s_-, s_+]$.

By countable stability of Hausdorff dimension $\xi(s)$, to prove (5.7), it is enough to show that a.s. $\mathcal{H}^\beta(\Theta^{s,u}(D_\eta) \cap B_d(0)) = 0$ for each $\beta > \xi(s) + o_u(1)$, and each $d \in (0, 1)$. It is moreover enough to prove the result restricted to the event $\mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)$ (in the notation of Theorem 4.1) for an arbitrary choice of $\mu \in \mathcal{M}$.

Let

$$r > \frac{1-s-u}{1-s+u}.$$

Note that we can take $r = 1 + o_u(1)$. For $n \in \mathbf{N}$ let $\mathcal{D}^n = 2^{-n(1-s)-4}\mathbf{Z}^2$ be the dyadic lattice of mesh size $2^{-n(1-s)-4}$. For $z \in \mathcal{D}^n$, let $B_0^n(z)$, $B_1^n(z)$, $B_2^n(z)$, and $B_3^n(z)$ be the disks centered at z of radii $2^{-n(1-s)-4}$, $2^{-n(1-s)-2}$, $2^{-n(1-s)+2}$, and $2^{-n(1-s)r+1}$, respectively.

Define Ψ_η as in Section 4.1. For $z \in \mathbf{D}$ let $E^n(z)$ be the event that the following occurs.

1. $\eta \cap B_2^n(z) \neq \emptyset$.
2. $\eta \cap B_1^n(z) = \emptyset$.
3. There is a $w \in B_0^n(z)$ with $2^{-n(s+2u)} \leq |\Psi'_\eta(w)| \leq 2^{-n(s-2u)}$.

On $E^n(z)$, we have

$$2^{-n(1-s)} \preceq \text{dist}(z, \partial D_\eta) \preceq 2^{-n(1-s)}, \quad 2^{-n(s+2u)} \preceq |\Psi'_\eta(z)| \preceq 2^{-n(s-2u)},$$

with constants uniform in $B_d(0)$ (the inequality for $|\Psi'_\eta|$ follows from the Koebe distortion theorem). So, by Proposition 4.1,

$$\mathbf{P}(E^n(z) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) \preceq 2^{-n(\gamma(s)-2\gamma_0(s)u)} \quad (5.15)$$

with constants uniform in $B_d(0)$.

Let \mathcal{U}^n be the set of disks $B_3^n(z)$ for $z \in \mathcal{D}^n$ such that $z \in B_d(0)$ and $E^n(z)$ occurs. Note that the cardinality of the set of disks which can belong to \mathcal{U}^n is of order $2^{2n(1-s)}$. We claim that $\Theta^{s,u}(D_\eta) \cap B_d(0) \subset \bigcup_{n \geq N} \bigcup_{B_3^n(z) \in \mathcal{U}^n} B_3^n(z)$ for each $N \in \mathbf{N}$.

Indeed, suppose $x \in \Theta^{s,u}(D_\eta) \cap B_d(0)$. By Lemma 5.5, for any $\delta > 0$, we can find a sequence $n_k \rightarrow \infty$ and a sequence of points $w_k \in D_\eta$ converging to x such that for each k , $2^{-n_k(1-s)-2} \leq \text{dist}(w_k, \partial D_\eta) \leq 2^{-n_k(1-s)}$, $|w_k - x| \leq 2^{-n_k(1-s)r}$, and $2^{-n_k(s+2u)} \leq |\Psi'_\eta(w_k)| \leq 2^{-n_k(s-2u)}$.

Each w_k belongs to $B_0^{n_k}(z)$ for some $z \in \mathcal{D}^{n_k}$. Our hypothesis on the distance from w_k to ∂D_η implies that conditions 1 and 2 in the definition of $E^{n_k}(z)$ hold for this z . Clearly, condition 3 also holds for this z . Thus for such a z , $E^n(z)$ holds and we have $x \in B_3^n(z)$ (here we use the condition on $|w_k - x|$). This proves our claim.

Thus, for any $n \in \mathbf{N}$, $\bigcup_{n \geq m} \mathcal{U}^n$ is a cover of $\Theta^{s,u}(\partial D_\eta) \cap B_d(0)$. Each set in this cover has diameter $\preceq 2^{-n(1-s)r}$ and we have by (5.15) that

$$\begin{aligned} \mathbf{E} \left(\mathbf{1}_{\mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)} \sum_{n=m}^{\infty} \sum_{U \in \mathcal{U}_m} (\text{diam } B^n(z))^\beta \right) &\preceq \sum_{n=m}^{\infty} \sum_{z \in \mathcal{D}^n \cap B_d(0)} 2^{-n\beta(1-s)r} \mathbf{P}(E^n(z) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\Psi_\eta^-, \mu)) \\ &\preceq \sum_{n=m}^{\infty} 2^{2n(1-s)} 2^{-n\beta(1-s)r} 2^{-n(\gamma(s)-2\gamma_0(s)u)}. \end{aligned} \quad (5.16)$$

This tends to 0 as $m \rightarrow \infty$ provided

$$\beta > 2 - \frac{\gamma(s) + 2\gamma_0(s)u}{(1-s)r} = \xi(s) + o_u(1),$$

where the $o_u(1)$ can be taken to be uniform for s in compact subsets of $(-1, 1)$. Since μ is arbitrary we conclude that $\mathcal{H}^\beta(\Theta^{s;u}(\partial D_\eta) \cap B_d(0)) = 0$ for any such β . \square

From Lemma 5.3, we can deduce the upper bounds on $\dim_{\mathcal{H}} \Theta^{s;\geq}(D_\eta)$ and $\dim_{\mathcal{H}}(\Theta^{s;\leq}(D_\eta))$ in Theorem 1.1.

Proposition 5.6. *Suppose we are in the setting of Theorem 1.1. Then a.s.*

$$\begin{aligned} \dim_{\mathcal{H}} \Theta^{s;\geq}(D_\eta) &\leq \xi(s), & \frac{\kappa}{4} &\leq s \leq s_+ \\ \dim_{\mathcal{H}} \Theta^{s;\leq}(D_\eta) &\leq \xi(s), & s_- &\leq s \leq \frac{\kappa}{4}. \end{aligned}$$

Proof. For $s \leq \kappa/4$ and any $n \in \mathbf{N}$, we have

$$\Theta^{s;\leq}(D_\eta) \subset \bigcup_{j=m_0}^{m_1} \Theta^{-j/n;1/n}(D_\eta), \quad (5.17)$$

where m_0 is the greatest integer such that m_0/n is smaller than s_- and m_1 is the least integer such that $m_1/n \geq s$. We have that $\xi(s)$ is increasing on $[0, \kappa/4]$ and $\gamma_0(s)$ is uniformly bounded for $s \in [s_-, s_+]$ (and for $s \leq s_1$ for some $s_1 < 1$ in the case $\kappa = 4$). Our upper bound in the case $s \leq \kappa/4$ thus follows from Lemma 5.3 and (5.17) together with stability of Hausdorff dimension under unions. Similarly for the case $s \geq \kappa/4$. \square

5.3 Upper bound for the integral means spectrum

In this subsection we will prove the upper bound for the bulk integral means spectrum of the SLE curve in Corollary 1.8. In light of Lemma 2.17, it will be enough to prove an upper bound for the bulk integral means spectrum of $\mathbf{D} \setminus \eta^t$ for given $t \geq 0$ in the case of an ordinary SLE_κ from $-i$ to i in \mathbf{D} for $\kappa \leq 4$.

Proposition 5.7. *Let $\kappa \in (0, 4]$ and let $\text{IMS}^*(a)$ be defined as in Corollary 1.8. Let η be a chordal SLE_κ from $-i$ to i in \mathbf{D} . For each $t > 0$ and each $a \in \mathbf{R}$, a.s. $\text{IMS}_{\mathbf{D} \setminus \eta^t}^{\text{bulk}}(a) \leq \text{IMS}^*(a)$.*

Proof. Let (f_t) be the centered Loewner maps for η , as defined in Section 3.5. For $\delta > 0$, let $U_t(\delta)$ be the set of $z \in \mathbf{D} \setminus \eta^t$ with $1 - |f_t^{-1}(z)| \geq \delta$ and $|z - i|, |z + i| \geq \delta$. Also define the sets $A_\epsilon^\zeta(f_t^{-1})$ be as in Section 1.3 (immediately following (1.10)). For any given $\zeta > 0$ there a.s. exists (random) $\delta > 0$ such that $A_\epsilon^\zeta(f_t^{-1}) \subset \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ for sufficiently small ϵ . Therefore, it is enough to show that for each $\delta > 0$ and each $\beta > \text{IMS}^*(a)$, we a.s. have

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a dz}{-\log \epsilon} \leq \beta. \quad (5.18)$$

Fix $\delta > 0$ and $\beta > \text{IMS}^*(a)$ as above. Also fix $t > 0$ and $c > 2$ and define the events $\overline{E}^{s;u}(z) = \overline{E}^{s;u}(z; t, c, 1 - \delta)$ as in Section 3.5 with $d = 1 - \delta$. Let s_- and s_+ be as in the statement of Theorem 1.1. For $n \in \mathbf{N}$ and $k \in \{0, \dots, n\}$, let

$$u_n = \frac{s_+ - s_-}{n}, \quad s_k^n = s_0 + k u_n.$$

For $n \in \mathbf{N}$ and $\epsilon > 0$, let $A_\epsilon^n(-)$ (resp. $A_\epsilon^n(+)$) be the set of $z \in \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ such that $|(f_t^{-1})'(z)| \leq \epsilon^{-s_-+1/n}$ (resp. $|(f_t^{-1})'(z)| \geq \epsilon^{-s_+-1/n}$). For $k \in \{0, \dots, n\}$, let $A_\epsilon^n(k)$ be the set of $z \in \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ such that the $\overline{E}^{s_k^n;u_n}(z)$ occurs. Let $\ell_\epsilon^n(k)$ be the Lebesgue measure of $A_\epsilon^n(k)$ and let $\ell_\epsilon^n(\pm)$ be the Lebesgue measure of $A_\epsilon^n(\pm)$.

In what follows, we require implicit constants to be independent of ϵ , but not of n or k , and we denote by $o_n(1)$ a term which tends to 0 as $n \rightarrow \infty$ and does not depend on k or ϵ .

By construction, we have $\partial B_{1-\epsilon}(0) \cap U_t(\delta) = A_\epsilon^n(-) \cup A_\epsilon^n(+) \cup \bigcup_{k=0}^n A_\epsilon^n$, whence

$$\int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a dz \preceq \sum_{k=0}^n \epsilon^{-as_k^n + o_n(1)} \ell_\epsilon^n(k) + \epsilon^{-as-} \ell_\epsilon^n(-) + \epsilon^{-as+} \ell_\epsilon^n(+).$$

The proof of Lemma 5.1 shows that for each $n \in \mathbf{N}$, there a.s. exists a random $\epsilon_0^n > 0$ such that for $\epsilon \in (0, \epsilon_0^n]$, the sets $A_\epsilon^n(-)$ and $A_\epsilon^n(+)$ are empty. Hence for $\epsilon \in (0, \epsilon_0^n]$, we have

$$\int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a dz \preceq \sup_{k \in \{0, \dots, n\}} \epsilon^{-as_k^n + o_n(1)} \ell_\epsilon^n(k). \quad (5.19)$$

By Corollary 3.11, for $k \in \{0, \dots, n\}$, we have

$$\mathbf{E}(\ell_\epsilon^n(k)) \preceq \epsilon^{\alpha(s_k^n) + o_n(1)}.$$

By Chebyshev's inequality,

$$\mathbf{P}\left(\epsilon^{-as_k^n} \ell_\epsilon^n(k) > \epsilon^{-\beta}\right) \preceq \epsilon^{\alpha(s_k^n) - as_k^n + \beta + o_n(1)}. \quad (5.20)$$

We have

$$\inf_{s \in [s_-, s_+]} (\alpha(s_k^n) - as_k^n) = -\text{IMS}^*(a). \quad (5.21)$$

Note that the range (a_-, a_+) in Corollary 1.8 is precisely the set of $a \in \mathbf{R}$ for which the minimizer in (5.21) is not equal to s_- or s_+ . It follows that for sufficiently large $n \in \mathbf{N}$, depending only on β , we have

$$\mathbf{P}\left(\sup_{k \in \{0, \dots, n\}} \epsilon^{-as_k^n} \ell_\epsilon^n(k) > \epsilon^{-\beta}\right) \preceq \epsilon^{\beta - \text{IMS}^*(a) + o_n(1)}.$$

Since $\beta > \text{IMS}^*(a)$, if $n \in \mathbf{N}$ is chosen sufficiently large (depending only on β and a), then the Borel-Cantelli lemma together with (5.19) implies that a.s.

$$\int_{\partial B_{1-2^{-j}}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a dz \leq 2^{-j\beta}$$

for sufficiently large $j \in \mathbf{N}$. By the Koebe distortion theorem, it follows that a.s.

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a dz}{-\log \epsilon} \leq \beta.$$

This proves (5.18), and hence the statement of the proposition. \square

6 Two point estimate

6.1 Event at the hitting time

In this subsection we introduce an event which will serve as the basic building block for the “perfect points” which we will use to prove our lower bounds on the Hausdorff dimensions of $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$.

6.1.1 Definition of the event

Suppose $\eta : [0, \infty] \rightarrow \overline{\mathbf{D}}$ is a random simple curve in \mathbf{D} which connects distinct points $x, y \in \partial \mathbf{D}$ and does not otherwise hit $\partial \mathbf{D}$. We recall the notation

$$\eta^t = \eta([0, t]), \quad \eta = \eta([0, \infty])$$

from Section 2.1.

Throughout this section, for $\beta > 0$, we will write

$$\mathcal{B}_\beta := B_{e^{-\beta}}(0).$$

Let $\bar{\eta}$ be the time reversal of η . Fix parameters $\beta > 0$, $a \in (0, 1/4)$, $u, c > 0$, $q \in (-1/2, \infty)$, and $\mu \in \mathcal{M}$. The parameter q corresponds to $s/(1-s)$, for s the parameter of Theorem 1.1.

Let $E = E_\beta^{q;u}(\eta; a, c, \mu)$ be the event that the following holds.

1. Let τ_β (resp. $\bar{\tau}_\beta$) be the first time that η (resp. $\bar{\eta}$) hits $\partial\mathcal{B}_\beta$. Then $\tau_\beta, \bar{\tau}_\beta < \infty$.
2. The harmonic measure from 0 in $\mathbf{D} \setminus (\eta^{\tau_\beta} \cup \bar{\eta}^{\bar{\tau}_\beta})$ of each of the two sides of η^{τ_β} and each of the two sides of $\bar{\eta}^{\bar{\tau}_\beta}$ is at least a .
3. Let $\phi_\beta : \mathbf{D} \setminus (\eta^{\tau_\beta} \cup \bar{\eta}^{\bar{\tau}_\beta}) \rightarrow \mathbf{D}$ be the unique conformal transformation which takes x to $-i$, y to i , and the midpoint m of $[x, y]_{\partial\mathbf{D}}$ to 1. Then we have that $c^{-1}e^{-\beta(q+u)} \leq |\phi'_\beta(0)| \leq ce^{-\beta(q-u)}$.
4. $\mathcal{G}'(\eta^{\tau_\beta} \cup \bar{\eta}^{\bar{\tau}_\beta}, \mu)$ occurs (Definition 2.6).

We will be primarily interested in the case where η is a chordal SLE_κ process from x to y . In this case, we write \mathcal{F}_β for the σ -algebra generated by $\eta|_{[0, \tau_\beta]}$ and $\bar{\eta}|_{[0, \bar{\tau}_\beta]}$.

6.1.2 Setup and upper bound

We need to estimate the probability of the event E of the preceding subsection. To this end we will prove the following.

Proposition 6.1. *Let $b > 0$ and let $x, y \in \partial\mathbf{D}$ with $|x - y| \geq b$. Let η be a chordal SLE_κ process from x to y in \mathbf{D} . Let $E = E_\beta^{q;u}(\eta; a, c, \mu)$ be as in Section 6.1. Let γ be as in (4.1) and let*

$$\gamma^*(q) := (q+1)\gamma\left(\frac{q}{1+q}\right) = \frac{8\kappa + 8\kappa q + (4-\kappa)^2 q^2}{8(\kappa + 2\kappa q)}. \quad (6.1)$$

There exists a continuous function $\gamma_0^ : (-1/2, \infty) \rightarrow (0, \infty)$ (with $\gamma_0^*(q)$ depending only on q) such that the following is true for sufficiently small $u > 0$ (depending only on q). We have*

$$\mathbf{P}(E) \leq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)u)}. \quad (6.2)$$

Moreover, there exists $\mu \in \mathcal{M}$ depending only on b such that for each $a \in (0, 1/4)$ and $u, c > 0$, there exists $\beta_0 > 0$ (depending on a, u, c and b) such that for $\beta \geq \beta_0$ and $z \in B_d(0)$,

$$\mathbf{P}(E) \geq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)u)}. \quad (6.3)$$

The implicit constants in (6.2) and 6.3 are independent of β and uniform for $x, y \in \partial\mathbf{D}$ with $|x - y| \geq b$.

First we will prove the upper bound (6.2), which is a straightforward consequence of Theorem 4.1.

Proof of Proposition 6.1, upper bound. Let $\hat{\eta}$ be the image under ϕ_β of the part of η lying between $\eta(\tau_\beta)$ and $\bar{\eta}(\bar{\tau}_\beta)$. Let $\hat{x} = \phi_\beta(\eta(\tau_\beta))$ and $\hat{y} = \phi_\beta(\bar{\eta}(\bar{\tau}_\beta))$, so that the conditional law of $\hat{\eta}$ given \mathcal{F}_β is that of an SLE_κ from \hat{x} to \hat{y} in \mathbf{D} . Let $C > 1$. Let $\hat{E} = \hat{E}(C)$ be the event that the following occurs.

1. $\hat{\eta}$ does not exit $\phi_\beta(\mathcal{B}_1)$.
2. Define the domain $D_{\hat{\eta}}$ as in Section 4.1. Then $\phi_\beta(0) \in D_{\hat{\eta}}$ and $C^{-1}(1 - |\phi_\beta(0)|) \leq \text{dist}(\phi_\beta(0), \partial D_{\hat{\eta}}) \leq C(1 - |\phi_\beta(0)|)$.
3. Let $\Phi_{\hat{\eta}} : D_{\hat{\eta}} \rightarrow \mathbf{D}$ be the conformal map fixing $-i$, i , and 1. Then $C^{-1} \leq |\Phi'_{\hat{\eta}}(\phi_\beta(0))| \leq C$.

It follows from condition 2 in the definition of E and [MW14, Lemma 2.3] that we can find a $C > 0$ depending only on a such that for sufficiently large β , $\mathbf{P}(\widehat{E}|E) \succeq 1$. Thus

$$\mathbf{P}(E) \asymp \mathbf{P}(E \cap \widehat{E}). \quad (6.4)$$

So, it will suffice to prove an upper bound for $\mathbf{P}(E \cap \widehat{E})$.

Let $s \in (-1, 1)$ and $\epsilon > 0$ be chosen so that

$$\frac{s}{1-s} = q, \quad \epsilon^{1-s} = e^{-\beta}. \quad (6.5)$$

Let D_η , Ψ_η , Ψ_η^- , and $\mathcal{E}_\epsilon^{s;u}(\eta, 0; c)$ be as in Section 4.1. It follows from Lemma 2.8 and condition 4 in the definition of E that

$$E \subset \mathcal{G}(\phi_\beta, \mu') \quad (6.6)$$

for some $\mu' \in \mathcal{M}$ depending only on μ . By combining this with condition 1 in the definition \widehat{E} we see that that $E \cap \widehat{E} \subset \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')$ for some (possibly smaller) $\mu' \in \mathcal{M}$ depending only on μ . We furthermore have

$$\Psi_\eta = \Psi_{\widehat{\eta}} \circ \phi_\beta.$$

Hence we have

$$E \cap \widehat{E} \subset \mathcal{E}_\epsilon^{s;u}(\eta, 0; c) \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu')$$

for suitable choice of μ' and c . Thus (6.2) follows from (6.4) and the upper bound in Theorem 4.1. Note that we can take the dependence on u to be linear (with slope depending on q) since the exponent in the upper bound in Theorem 4.1 depends smoothly on $s \in (-1, 1)$ and $u > 0$ sufficiently small. \square

6.1.3 Lower bound

The proof of the lower bound in Proposition 6.1 will take substantially more work than the proof of the upper bound. The basic idea is to stop η and $\bar{\eta}$ at times t_0 and \bar{t}_0 for which the following is true. On the event $\mathcal{E}_\beta^{s;u}(\cdot)$ of Theorem 4.1, the conformal map from $\mathbf{D} \setminus (\eta^{t_0} \cup \bar{\eta}^{\bar{t}_0})$ to \mathbf{D} which takes x to $-i$, y to i , and the midpoint of $[x, y]_{\partial\mathbf{D}}$ to 1 has the same derivative behavior as the conformal map $\Psi_\eta : \mathbf{D} \setminus \eta \rightarrow \mathbf{D}$ with the same normalization; the points $\eta(t_0)$ and $\bar{\eta}(\bar{t}_0)$ are at distance slightly less than $e^{-\beta}$ from 0; and the conditional law of the remainder of the curve given $\eta^{t_0} \cup \bar{\eta}^{\bar{t}_0}$ is that of a chordal SLE $_\kappa$. We also need to require that $\eta(t_0)$ and $\bar{\eta}(\bar{t}_0)$ are sufficiently far apart in a conformal sense, so that they do not immediately link up after times t_0 and \bar{t}_0 . We then condition on $\eta^{t_0} \cup \bar{\eta}^{\bar{t}_0}$ and use standard arguments to get that the curves reach \mathcal{B}_β without any pathological behavior. The main difficulty in the proof is constructing the times t_0 and \bar{t}_0 .

We start by inductively defining a means of growing η and $\bar{\eta}$ simultaneously to get an increasing family of hulls $K_t \subset \mathbf{D}$. Assume η (resp. $\bar{\eta}$) is parametrized in such a way that its image under the conformal map $\mathbf{D} \rightarrow \mathbf{H}$ taking $-i$ to 0, i to ∞ , and 0 to i (resp. the reciprocal of this conformal map) is parametrized by half plane capacity. Let σ_1 be the first time t that $\text{hm}^0(\eta^t; \mathbf{D} \setminus \eta^t) = 1/2$. This time is a.s. finite since a Brownian motion started from 0 has probability at least 1/2 to hit η before $\partial\mathbf{D}$. For $t \leq \sigma_1$, let $K_t = \eta^t$. Let $\bar{\sigma}_1$ be the first \bar{t} that either $\text{hm}^0(\bar{\eta}^{\bar{t}}; \mathbf{D} \setminus (\eta^{\sigma_1} \cup \bar{\eta}^{\bar{t}})) = 1/2$ or $\bar{\eta}(\bar{t}) = \eta(\sigma_1)$. For $t \in [\sigma_1, \sigma_1 + \bar{\sigma}_1]$ let $K_t = \eta^{\sigma_1} \cup \bar{\eta}^{t-\sigma_1}$.

Inductively, suppose $n \geq 2$ and σ_{n-1} , $\bar{\sigma}_{n-1}$, and K_t for $t \leq \sigma_{n-1} + \bar{\sigma}_{n-1}$ have been defined. If $K_{\sigma_{n-1} + \bar{\sigma}_{n-1}} = \eta$ we let $\sigma_n = \sigma_{n-1}$ and $\bar{\sigma}_n = \bar{\sigma}_{n-1}$. Otherwise, let σ_n be the least $t \geq \sigma_{n-1}$ such that either $\text{hm}^0(\eta^t; \mathbf{D} \setminus (\eta^t \cup \bar{\eta}^{\bar{\sigma}_{n-1}})) = 1/2$ or $\eta(t) = \bar{\eta}(\bar{\sigma}_{n-1})$. Let $K_t = \eta^{t-\bar{\sigma}_{n-1}} \cup \bar{\eta}^{\bar{\sigma}_{n-1}}$ for $t \in [\sigma_{n-1} + \bar{\sigma}_{n-1}, \sigma_n + \bar{\sigma}_{n-1}]$. Let $\bar{\sigma}_n$ be the first $\bar{t} \geq \bar{\sigma}_{n-1}$ such that either $\text{hm}^0(\bar{\eta}^{\bar{t}}; \mathbf{D} \setminus (\eta^{\sigma_n} \cup \bar{\eta}^{\bar{t}})) = 1/2$ or $\bar{\eta}(\bar{t}) = \eta(\sigma_n)$. Let $K_t = \eta^{\sigma_n} \cup \bar{\eta}^{t-\sigma_n}$ for $t \in [\sigma_n + \bar{\sigma}_{n-1}, \sigma_n + \bar{\sigma}_n]$.

For each $t \geq 0$, let T_t (resp. \bar{T}_t) be the time such that $\eta(T_t)$ (resp. $\bar{\eta}(\bar{T}_t)$) is the tip of the part of η (resp. $\bar{\eta}$) included in K_t . Observe that the Markov property and reversibility of SLE imply that for each t , the conditional law of $\eta \setminus K_t$ given K_t is that of a chordal SLE $_\kappa$ from $\eta(T_t)$ to $\bar{\eta}(\bar{T}_t)$ in $\mathbf{D} \setminus K_t$.

Lemma 6.2. *Let $\sigma_\infty = \lim_{n \rightarrow \infty} \sigma_n$ and $\bar{\sigma}_\infty = \lim_{n \rightarrow \infty} \bar{\sigma}_n$ (the limits necessarily exist by monotonicity). Let $K_\infty = \eta^{\sigma_\infty} \cup \bar{\eta}^{\bar{\sigma}_\infty}$. Then we a.s. have*

$$\lim_{n \rightarrow \infty} \text{hm}^0(\eta^{\sigma_n}; \mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_n}) = \lim_{n \rightarrow \infty} \text{hm}^0(\eta^{\sigma_n}; \mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_{n-1}}) = \text{hm}^0(\eta^{\sigma_\infty}; \mathbf{D} \setminus K_\infty)$$

and

$$\lim_{n \rightarrow \infty} \text{hm}^0(\bar{\eta}^{\bar{\sigma}_n}; \mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_n}) = \lim_{n \rightarrow \infty} \text{hm}^0(\bar{\eta}^{\bar{\sigma}_{n-1}}; \mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_{n-1}}) = \text{hm}^0(\bar{\eta}^{\bar{\sigma}_\infty}; \mathbf{D} \setminus K_\infty).$$

Proof. We a.s. have $0 \notin \eta$ so it is a.s. the case that for each $\epsilon > 0$, we can find a random $\delta > 0$ such that for any $z \in \eta$, the probability that a Brownian motion started from 0 hits $B_\delta(z)$ before leaving \mathbf{D} is at most ϵ . By a.s. continuity of η , we can a.s. find a (random) $N \in \mathbf{N}$ such that for $n \geq N$, $\eta([\sigma_n, \sigma_\infty]) \subset B_\delta(\eta(\sigma_\infty))$ and $\bar{\eta}([\bar{\sigma}_n, \bar{\sigma}_\infty]) \subset B_\delta(\bar{\eta}(\bar{\sigma}_\infty))$. Hence with probability at least $1 - \epsilon$, a Brownian motion started from 0 exists $\mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_n}$ at the same place it exists $\mathbf{D} \setminus K_\infty$. This proves the limits involving $K_{\sigma_n + \bar{\sigma}_n}$. The limits involving $K_{\sigma_n + \bar{\sigma}_{n-1}}$ are proven similarly. \square

Lemma 6.3. *We a.s. have $K_\infty = \eta$. Let $z_\infty = \eta(\sigma_\infty) = \bar{\eta}(\bar{\sigma}_\infty)$ be the meeting point. On the event that 0 lies to the right of η and $\text{dist}(0, \eta) \leq e^{-\beta}$, it holds a.s. that $\text{hm}^0(\eta^{\sigma_\infty}; D_\eta)$ and $\text{hm}^0(\bar{\eta}^{\bar{\sigma}_\infty}; D_\eta)$ are each at least $1/2 - o_\beta(1)$, where the $o_\beta(1)$ is a deterministic quantity which tends to 0 as $\beta \rightarrow 0$.*

Proof. First we argue that $K_\infty = \eta$. Suppose not. Almost surely, either $\text{hm}^0(\eta^{\sigma_\infty}; \mathbf{D} \setminus K_\infty)$ or $\text{hm}^0(\bar{\eta}^{\bar{\sigma}_\infty}; \mathbf{D} \setminus K_\infty)$ is $< 1/2$. Suppose $\text{hm}^0(\eta^{\sigma_\infty}; \mathbf{D} \setminus K_\infty) < 1/2$. The other case is treated similarly. By Lemma 6.2 we a.s. have $\text{hm}^0(\eta^{\sigma_n}; \mathbf{D} \setminus K_{\sigma_n + \bar{\sigma}_{n-1}}) < 1/2$ for sufficiently large n . By definition of σ_n this can be the case only if $\eta(\sigma_n) = \bar{\eta}(\bar{\sigma}_{n-1})$ which implies $K_\infty = \eta$.

It is immediate from Lemma 6.2 that $\text{hm}^0(\eta^{\sigma_\infty}; D_\eta)$ and $\text{hm}^0(\bar{\eta}^{\bar{\sigma}_\infty}; D_\eta)$ are each at most $1/2$. Furthermore, the Beurling estimate implies $\text{hm}^0(\partial \mathbf{D}; D_\eta) = o_\beta(1)$. Hence

$$\text{hm}^0(\eta^{\sigma_\infty}; D_\eta) = 1 - \text{hm}^0(\bar{\eta}^{\bar{\sigma}_\infty}; D_\eta) - \text{hm}^0(\partial \mathbf{D}; D_\eta) \geq 1/2 - o_\beta(1)$$

and similarly for $\bar{\eta}^{\bar{\sigma}_\infty}$. \square

Lemma 6.4. *For $t \geq 0$, let Φ_t be the conformal map from the connected component of $\mathbf{D} \setminus K_t$ with $[x, y]_{\partial \mathbf{D}}$ on its boundary to \mathbf{D} taking x^+ to $-i$, y^- to i , and the midpoint m of $[x, y]_{\partial \mathbf{D}}$ to 1. Let $\tilde{\Phi}_t$ be the conformal map from this same connected component to \mathbf{D} which fixes 0 and takes m to 1. Let Ψ_η be as in Section 4.1. Let $\mu \in \mathcal{M}$. There is a $C > 1$ and a $\beta_0 > 0$, depending only on μ and b , such that the following is true. If $\beta \geq \beta_0$ then on the event $\mathcal{G}(\Psi_\eta, \mu) \cap \{\text{dist}(0, \eta) \leq e^{-\beta}\} \cap \{0 \in D_\eta\}$, there a.s. exists a time $\tau > 0$ such that the following is true.*

1. $\text{dist}(0, K_\tau) \leq C \text{dist}(0, \eta)$.
2. $C^{-1} |\Psi'_\eta(0)| \leq |\Phi'_\tau(0)| \leq C |\Psi'_\eta(0)|$.
3. $\tilde{\Phi}_\tau(\eta(T_\tau))$ and $\tilde{\Phi}_\tau(\bar{\eta}(\bar{T}_\tau))$ lie in $[i, -i]_{\partial \mathbf{D}}$.
4. $\text{hm}^0(\eta \setminus K_\tau; D_\eta) \geq 1/4 + o_\beta(1)$, with the $o_\beta(1)$ deterministic and depending only on β .

Proof. Throughout, we assume we are working on the event $\mathcal{G}(\Psi_\eta, \mu) \cap \{\text{dist}(0, \eta) \leq e^{-\beta}\} \cap \{0 \in D_\eta\}$ and we require all implicit constants to be deterministic and depend only on μ .

Let $\tilde{\Psi}_\eta : D_\eta \rightarrow \mathbf{D}$ be the conformal map which fixes 0 and takes m to 1. If z_∞ is as in Lemma 6.3 then by conformal invariance of harmonic measure we have

$$|\tilde{\Psi}_\eta(z_\infty) + 1| = o_\beta(1), \tag{6.7}$$

at a deterministic rate.

Let τ be the first time t that $\tilde{\Psi}_\eta(\eta(T_t))$ and $\tilde{\Psi}_\eta(\bar{\eta}(\bar{T}_t))$ are both in $[i, -i]_{\partial \mathbf{D}}$. By Lemma 6.3 such a t necessarily exists provided β is at least some universal constant. Let $\tilde{A}_\tau = [\tilde{\Psi}_\eta(\bar{\eta}(\bar{T}_\tau)), \tilde{\Psi}_\eta(\eta(T_\tau))]_{\partial \mathbf{D}}$. By continuity one of the two endpoints of \tilde{A}_τ is $-i$ or i so by (6.7) we have $\text{hm}^0(\tilde{A}_\tau; \mathbf{D}) \geq 1/4 - o_\beta(1)$. Furthermore, the harmonic measure from 0 in \mathbf{D} of each of the two arcs connecting \tilde{A}_τ and 1 is at least $1/4 - o_\beta(1)$.

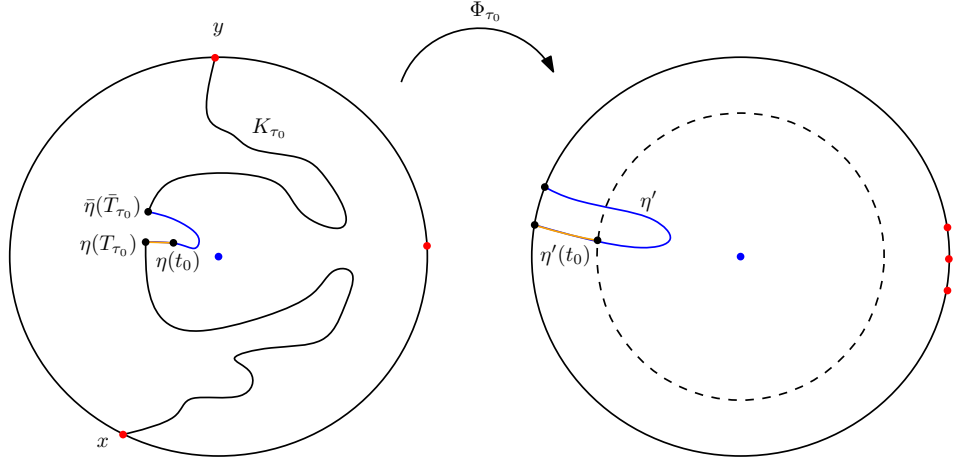


Figure 6.1: An illustration of the argument of Lemma 6.5 in the case $\{T' < \infty\}$. The hull K_{τ_0} is shown in black. the curve η' and its pre-image under $\tilde{\Phi}_{\tau_0}$ are shown in blue. The extra part of the curve which we grow after growing K_{τ_0} is shown in orange.

Let $A_\tau = \tilde{\Psi}_\eta^{-1}(\tilde{A}_\tau) = \eta \setminus K_\tau$. By conformal invariance of harmonic measure we have that $\text{hm}^0(\eta^{T_t}; D_\eta)$, $\text{hm}^0(\bar{\eta}^{\bar{T}_t}; D_\eta)$, and $\text{hm}^0(A_\tau; D_\eta)$ are each at least $1/4 - o_\beta(1)$. By Lemma B.4 (applied with $I = [x, y]_{\partial \mathbf{D}}$ and $\phi = \Phi_\tau$) we have $\text{dist}(0, K_\tau) \asymp \text{dist}(0, \eta)$ and $|\Phi'_\tau(0)| \asymp |\Psi'_\eta(0)|$. Since $\tilde{\Psi}_\eta(\eta(T_\tau))$ and $\tilde{\Psi}_\eta(\bar{\eta}(\bar{T}_\tau))$ lie in $[i, -i]_{\partial \mathbf{D}}$ and removing A_τ can only increase the harmonic measure from 0 of parts of ∂D_η outside of A_τ , we find that $\tilde{\Phi}_\tau(\eta(T_\tau))$ and $\tilde{\Phi}_\tau(\bar{\eta}(\bar{T}_\tau))$ must lie in $[i, -i]_{\partial \mathbf{D}}$. Thus, the conditions of the lemma hold for this choice of τ . \square

Lemma 6.5. *Let $v > 0$, $\zeta > 0$, and $\mu_0 \in \mathcal{M}$. For $\beta > 0$ and two times $t, \bar{t} > 0$, let $E_\beta^0(t, \bar{t}) = E_\beta^0(t, \bar{t}; v, \zeta, \mu_0)$ be the event that the following occurs.*

1. $32e^{-\beta} \leq \text{dist}(0, \eta^t \cup \bar{\eta}^{\bar{t}}) \leq e^{-\beta(1-v)}$.
2. Let $\phi_{t, \bar{t}} : \mathbf{D} \setminus (\eta^t \cup \bar{\eta}^{\bar{t}}) \rightarrow \mathbf{D}$ be the conformal map which takes x^+ to $-i$, y^- to i , and the midpoint m of $[x, y]_{\partial \mathbf{D}}$ to 1. Then $e^{-\beta(q+v)} \leq |\phi'_{t, \bar{t}}(0)| \leq e^{-\beta(q-v)}$.
3. Let $\psi_{t, \bar{t}} : \mathbf{D} \setminus (\eta^t \cup \bar{\eta}^{\bar{t}}) \rightarrow \mathbf{D}$ be the conformal map which fixes 0 and takes m to 1. Then $|\psi_{t, \bar{t}}(\eta(t)) - \psi_{t, \bar{t}}(\bar{\eta}(\bar{t}))| \geq \zeta$.
4. $\mathcal{G}'(\eta^t \cup \bar{\eta}^{\bar{t}}, \mu_0)$ occurs.

There is a deterministic $\zeta > 0$ and $\mu_0 \in \mathcal{M}$, independent of v and β , such that for each $v > 0$, there exists $\beta_0 > 0$ such that for each $\beta \geq \beta_0$, there exist random times t_0 and \bar{t}_0 such that

$$\mathbf{P}(E_\beta^0(t_0, \bar{t}_0)) \geq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v)}, \quad (6.8)$$

where here $\gamma^*(q)$ and $\gamma_0^*(q)$ are as in Proposition 6.1 and the implicit constant is independent of β and uniform in x, y with $|x - y|$ bounded below. Furthermore, we can choose t_0 and \bar{t}_0 in such a way that the conditional law given $\eta^{t_0} \cup \bar{\eta}^{\bar{t}_0}$ of the part of η between $\eta(t_0)$ and $\bar{\eta}(\bar{t}_0)$ on the event $E_\beta^0(t_0, \bar{t}_0)$ is that of a chordal SLE $_\kappa$ from $\eta(t_0)$ to $\bar{\eta}(\bar{t}_0)$ in $\mathbf{D} \setminus (\eta^{t_0} \cup \bar{\eta}^{\bar{t}_0})$.

Proof. Fix $v' \in (0, v/4)$ and let $s = s(v') \in (-1, 1)$ and $\epsilon = \epsilon(s, v', \beta) > 0$ be chosen so that

$$s = \frac{q}{q+1} + o_{v'}(1), \quad \epsilon^{1-s} = e^{-\beta(1+o_{v'}(1))}, \quad \text{and} \quad \epsilon^{1-s+2v'} \geq 32e^{-\beta}.$$

Let $c > 0$ and let $\mathcal{E}_\epsilon^{s;v'}(\eta, 0; c)$ be the event of Section 4.1 (with v' in place of u). Let $\Psi_\eta : D_\eta \rightarrow \mathbf{D}$ and $\Psi_\eta^- : D_\eta^- \rightarrow \mathbf{D}$ be as in that subsection. Let $\mu' \in \mathcal{M}$ and let

$$\mathcal{E} := \mathcal{E}_\epsilon^{s;v'}(\eta, 0; c) \cap \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu').$$

By Theorem 4.1, if the parameter μ' is chosen appropriately then we can find $\beta_0 > 0$ as in the statement of the lemma such that

$$\mathbf{P}(\mathcal{E}) \succeq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v')},$$

for an appropriate choice of $\gamma_0^*(q)$ as in Proposition 6.1. Lemma 2.8 implies that we can find $\mu_0 \in \mathcal{M}$ depending only on μ' such that

$$\bigcup_{t, \bar{t} \geq 0} \mathcal{G}'(\eta^t \cup \bar{\eta}^{\bar{t}}, \mu_0) \subset \mathcal{G}(\Psi_\eta, \mu') \cap \mathcal{G}(\Psi_\eta^-, \mu'). \quad (6.9)$$

Let τ_0 be the first time τ that the first two conditions in the definition of $E_\beta^0(T_\tau, \bar{T}_\tau)$ are satisfied and that $\tilde{\Phi}_\tau(\eta(T_\tau))$ and $\tilde{\Phi}_\tau(\bar{\eta}(\bar{T}_\tau))$ (as defined just above Lemma 6.2) both lie in $[i, -i]_{\partial \mathbf{D}}$. By Lemma 6.4 and the definition of \mathcal{E} , if c is chosen sufficiently large then $\tau_0 < \infty$ a.s. on \mathcal{E} . Moreover, decreasing τ only increases $\text{hm}^0(\eta \setminus K_\tau; D_\eta)$, so on \mathcal{E} we a.s. have

$$\text{hm}^0(\eta \setminus K_{\tau_0}; D_\eta) \geq 1/4 - o_\beta(1). \quad (6.10)$$

Let $\eta' = \tilde{\Phi}_{\tau_0}(\eta \setminus K_{\tau_0})$, with the parametrization it inherits from η . By the strong Markov property, the conditional law of η' given K_{τ_0} is that of a chordal SLE $_\kappa$ from $x' := \tilde{\Phi}_{\tau_0}(\eta(T_{\tau_0}))$ to $y' := \tilde{\Phi}_{\tau_0}(\bar{\eta}(\bar{T}_{\tau_0}))$ in \mathbf{D} (here we used that we made τ_0 the *smallest* time for which our desired conditions are satisfied).

The definition of $E_\beta^0(t_0, \bar{t}_0)$ almost holds with $t_0 = T_{\tau_0}$ and $\bar{t}_0 = \bar{T}_{\tau_0}$, but $\tilde{\Phi}_{\tau_0}(\eta(T_{\tau_0}))$ and $\tilde{\Phi}_{\tau_0}(\bar{\eta}(\bar{T}_{\tau_0}))$ may be too close together. To this end, we will choose slightly larger times at which the images of the tips of η and $\bar{\eta}$ are separated. Note that (6.10) implies $\text{diam } \eta' \geq \zeta_0$ on \mathcal{E} for some universal constant $\zeta_0 \in (0, 1/4)$. Let $\bar{\eta}'$ be the time reversal of η' , with the parametrization it inherits from $\bar{\eta}$.

Let T' (resp. \bar{T}') be the first time that η' (resp. $\bar{\eta}'$) enters $B_{1-\zeta_0/4}(0)$. Let T'' be the first time $t \geq T_{\tau_0}$ that $\arg \eta'(t) \geq \arg x' + \zeta_0/8$. Let \bar{T}'' be the first time $\bar{t} \geq \bar{T}_{\tau_0}$ that $\arg \bar{\eta}'(\bar{t}) \leq \arg y' - \zeta_0/8$. Since $\text{diam } \eta' \geq \zeta_0$ a.s. on \mathcal{E} , either $|x' - y'| \geq \zeta_0/8$ or one of T', T'' or \bar{T}'' is finite on this event (if not, then η' is contained in the wedge $\{z \in \mathbf{D} : \arg y' - \zeta_0/8 \leq \arg z \leq \arg x' + \zeta_0/8, |z| \geq 1 - \zeta_0/8\}$ and this wedge has diameter $< \zeta_0$). Hence the intersection with \mathcal{E} of at least one of the events $\{|x' - y'| \geq \zeta_0/8\}$, $\{T' < \infty\}$, $\{T'' < T'\}$, or $\{\bar{T}'' < \bar{T}'\}$ has probability at least $\frac{1}{4}\mathbf{P}(\mathcal{E}) \succeq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v')}$.

It is therefore enough to show that the conclusion of the lemma is true in each of the four possible cases (provided β is sufficiently large). We will do this by choosing t_0 to be one of T_{τ_0}, T' , or T'' and \bar{t}_0 to be one of $\bar{T}_{\tau_0}, \bar{T}'$, or \bar{T}'' . By the strong Markov property, the last statement of the lemma holds for any such choice. Clearly, condition 1 in the definition of E_0 holds a.s. on \mathcal{E} for any such choice of t_0 and \bar{t}_0 and any $v' \in (0, v)$. By (6.9), condition 4 holds for any such choice. By conditions 1 and 2 in Lemma 6.4, on \mathcal{E} , we have

$$e^{-2\beta v'} \preceq \frac{|\Phi'_{\tau_0}(0)|}{|\Psi'_\eta(0)|} \preceq e^{2\beta v'} \quad \text{and} \quad 1 \leq \frac{\text{dist}(0, K_{\tau_0})}{\text{dist}(0, \eta)} \preceq e^{2\beta v'}$$

with deterministic, β -independent proportionality constants. By combining this with Lemma B.1 and condition 4 (c.f. Remark B.2), we infer that on \mathcal{E} ,

$$1 \leq \frac{\text{hm}^0(I; D \setminus K_{\tau_0})}{\text{hm}^0(I; D \setminus \eta)} \preceq e^{4\beta v'}. \quad (6.11)$$

for I a sub-arc of $[x, y]_{\partial \mathbf{D}}$ which is slightly smaller than $[x, y]_{\partial \mathbf{D}}$. For any choice of t_0 and \bar{t}_0 as above, we have $K_{\tau_0} \subset (\eta')^{t_0} \cup (\bar{\eta}')^{\bar{t}_0}$. Since $4v' < v$, (6.11) and a second application of Lemma B.1 yield condition 2 for large enough β .

Finally, we will verify that condition 3 holds in each of the four cases (for an appropriate choice of $\zeta > 0$ depending only on ζ_0). Here we note that $|x' - y'|$ is proportional to the harmonic measure from 0 of the boundary arc of $\mathbf{D} \setminus ((\eta')^{t_0} \cup (\bar{\eta}')^{\bar{t}_0})$ separating $\eta'(t_0)$ from $\bar{\eta}'(\bar{t}_0)$.

1. If $\mathbf{P}(|x' - y'| \geq \zeta_0/8, \mathcal{E}) \succeq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v')}$ then we can just set $t_0 = T_{\tau_0}$, $\bar{t}_0 = \bar{T}_{\tau_0}$, and $\zeta = \zeta_0/8$.
2. If $\mathbf{P}(T' < \infty, \mathcal{E}) \succeq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v')}$ then we set $t_0 = T'$ and $\bar{t}_0 = \bar{T}_{\tau_0}$. A Brownian motion has probability at least a constant $\zeta > 0$ depending only on ζ_0 to exit $B_{1-\zeta_0/16}(0)$ within distance $\zeta_0/4$ of 1 and then make a counterclockwise loop around the origin before leaving $\mathbf{D} \setminus B_{1-\zeta_0/16}(0)$. In this case it necessarily exits $\mathbf{D} \setminus (\eta')^{T'}$ on the left side of $(\eta')^{T'}$. See Figure 6.1.3 for an illustration in this case.
3. If $\mathbf{P}(T'' < T', \mathcal{E}) \succeq e^{-\beta(\gamma^*(q) + \gamma_0^*(q)v')}$ then we set $t_0 = T' \wedge T''$ and $\bar{t}_0 = \bar{T}_{\tau_0}$. A Brownian motion has probability at least a constant $\zeta > 0$ depending only on ζ_0 to exit \mathbf{D} before hitting any point outside of $\mathbf{D} \setminus B_{1-\zeta_0/8}(0)$ whose argument is not between $\arg x'$ and $\arg x' + \zeta_0/8$. If this is the case and $T' \leq T''$, then a Brownian motion necessarily exits $\mathbf{D} \setminus (\eta')^{T' \wedge T''}$ on the left side of $(\eta')^{T' \wedge T''}$.
4. The case for $\{\bar{T}'' < \bar{T}'\}$ is treated in the same manner as the case for $\{T'' < T'\}$.

Thus we have exhausted all possible cases and we conclude that condition 1 holds. \square

Proof of Proposition 6.1, lower bound. Suppose $\zeta > 0$, $\mu_0 \in \mathcal{M}$, and random times t_0, \bar{t}_0 are chosen so that the conclusion of Lemma 6.5 holds. Let $v > 0$ and let $\beta_0 > 0$ be chosen as in Lemma 6.5. Let $\beta > \beta_0$ and let $E_\beta^0 = E_0(t_0, \bar{t}_0, \zeta, \mu_0)$ be as in Lemma 6.5.

Let $\tilde{\beta} = -\log \text{dist}(0, \eta^{t_0} \cup \bar{\eta}^{\bar{t}_0})$. Note that on E_β^0 ,

$$\beta(1-v) \leq \tilde{\beta} \leq \beta - \log 32.$$

Fix $r \in (\log 16, \log 32)$. Let η_1 be the image under ψ_{t_0, \bar{t}_0} (defined as in Lemma 6.5) of the part of η between $\eta(t_0)$ and $\bar{\eta}(\bar{t}_0)$. Let x_1 and y_1 be its endpoints. Let τ'_1 (resp. $\bar{\tau}'_1$) be the first time η_1 (resp. $\bar{\eta}_1$) hits $\psi_{t_0, \bar{t}_0}(\mathcal{B}_{\tilde{\beta}+r})$. Let G_1 be the event that

1. $|\eta_1(\tau'_1) - \bar{\eta}_1(\bar{\tau}'_1)| \geq (1/32)e^{-r}$.
2. $\eta_1^{\tau'_1} \cup \bar{\eta}_1^{\bar{\tau}'_1} \subset \psi_{t_0, \bar{t}_0}(\mathcal{B}_1)$.
3. $\eta_1^{\tau'_1} \cup \bar{\eta}_1^{\bar{\tau}'_1}$ is disjoint from the $\zeta/2$ -neighborhood of the segment connecting 0 and the midpoint of the shorter arc between x_1 and y_1 .

By the Koebe quarter theorem we have

$$\mathcal{B}_{r+\log 16} \subset \psi_{t_0, \bar{t}_0}(\mathcal{B}_{\tilde{\beta}+r}) \subset \mathcal{B}_{r-\log 16}.$$

Hence by [MW14, Lemma 2.3], condition 3 in the definition of E_0 , and the last statement of Lemma 6.5 we have that $\mathbf{P}(G_1|E_0)$ is at least a β -independent positive constant.

For $k = 1, 2, 3, \dots$, let $\tilde{\psi}_k$ be the map from $\mathbf{D} \setminus (\eta^{\tau_{\tilde{\beta}+kr}} \cup \bar{\eta}^{\bar{\tau}_{\tilde{\beta}+kr}})$ to \mathbf{D} with $\tilde{\psi}_k(0) = 0$ and $\tilde{\psi}'_k(0) > 0$. For $k \geq 2$, let η_k be the image under $\tilde{\psi}_{k-1}$ of the part of η which lies between $\eta(\tau_{\tilde{\beta}+(k-1)r})$ and $\bar{\eta}(\bar{\tau}_{\tilde{\beta}+(k-1)r})$. The law of η_k given $\mathcal{F}_{\tilde{\beta}+(k-1)r}$ is that of a chordal SLE $_{\kappa}$ from $x_k := \tilde{\psi}_{k-1}(\eta(\tau_{\tilde{\beta}+(k-1)r}))$ to $y_k := \tilde{\psi}_{k-1}(\bar{\eta}(\bar{\tau}_{\tilde{\beta}+(k-1)r}))$. Let $\bar{\eta}_k$ be the time reversal of η_k .

Let τ'_k and $\bar{\tau}'_k$ be the hitting times of $\tilde{\psi}_{k-1}(\mathcal{B}_{\tilde{\beta}+kr})$ by η_k and $\bar{\eta}_k$, respectively. Fix $\delta > 0$ and for $k \geq 1$ let G_k be the event that η^{τ_k} (resp. $\bar{\eta}^{\bar{\tau}_k}$) is contained in the δ -neighborhood of the segment $[x_k, 0]$ (resp. $[y_k, 0]$).

By the Koebe quarter theorem, whenever $\tilde{\psi}_{k-1}$ is defined we have

$$\mathcal{B}_{r+\log 16} \subset \tilde{\psi}_{k-1}(\mathcal{B}_{\tilde{\beta}+kr}) \subset \mathcal{B}_{r-\log 16}.$$

By conformal invariance of harmonic measure, on G_{k-1} for $k \geq 2$, $|x_k - y_k|$ is at least a universal constant provided δ is taken sufficiently small. It now follows from [MW14, Lemma 2.3] that for each $k \geq 2$, $\mathbf{P}(G_k|G_{k-1}) \geq p$ for some $p > 0$ which depends only on δ .

Let k_* be the least integer k such that $kr + \tilde{\beta} \geq \beta$. Note that $k_* \leq \beta v/r$. Let

$$G^* := \bigcap_{k=1}^{k_*} G_k.$$

It is clear that on the event $E_0 \cap G^*$, conditions 1, 2, and 4 in the definition of E hold provided we take δ sufficiently small, depending on a .

It remains to deal with condition 3. For $k \geq 1$, let $\hat{\eta}_k$ be the curve obtained by connecting $\eta(\tau_{\tilde{\beta}+kr}^*)$ and $\bar{\eta}(\bar{\tau}_{\tilde{\beta}+kr}^*)$ via the arc of $\mathcal{B}_{\tilde{\beta}+kr}^*$ which does not disconnect 0 from $[x_*, y_*]_{\partial \mathbf{D}}$. Let $\Psi_{\hat{\eta}_k}$ be the conformal map from the connected component of $\mathbf{D} \setminus \hat{\eta}_k$ containing $[x_*, y_*]_{\partial \mathbf{D}}$ on its boundary to \mathbf{D} which takes x_* to $-i$, y_* to i , and the midpoint of $[x_*, y_*]_{\partial \mathbf{D}}$ to 1. By Lemma B.4, we have

$$C^{-1}|\Psi'_{\hat{\eta}_k}(0)| \leq |\phi'_{\beta'}(0)| \leq C|\Psi'_{\hat{\eta}_k}(0)|, \quad \forall \beta' \in [\tilde{\beta} + (k-1)r, \tilde{\beta} + kr], \quad \forall k \geq 2.$$

on G^* , for some deterministic $C > 0$ depending only on a , r , and μ . A similar statement holds for $k = 1$ provided we replace C with a constant $C_1 > 0$ which is allowed to depend on ζ (and hence on v), but not β .

In particular, on G^* we have

$$C_1^{-1}C^{-\beta v/r}e^{-\beta(q+v)} \leq |\phi'_{\beta}(0)| \leq C_1^{-1}C^{\beta v/r}e^{-\beta(q-v)}.$$

If we choose v such that $v \leq u/2$ and $C^{v/r} \leq e^{u/2}$ and c_1 sufficiently small, then condition 3 in the definition of E holds on $E_0 \cap G^*$. By Lemma 6.5 and our choice of parameters above we then have

$$\mathbf{P}(E) \geq \mathbf{P}(G_1|E_0)p^{k_*-1}e^{-\beta(\gamma^*(q)+v)} \succeq e^{-\beta(\gamma^*(q)+\gamma_0^*(q)u)}.$$

□

6.2 Events for the perfect points

In this subsection we will define events $E_{z,j}$ for each $z \in \mathbf{D}$ and $j \in \mathbf{N}$ which we will eventually use to construct subsets of $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$ (called the “perfect points”) whose Hausdorff dimension can be bounded below. The definition of the events $E_{z,j}$ involves a number of auxiliary objects which we list below.

Let $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$ and let $\lambda = \pi/\sqrt{\kappa}$ be as in (2.20). Let h be a zero boundary GFF on \mathbf{D} plus a harmonic function chosen in such a way that if $\psi : \mathbf{H} \rightarrow \mathbf{D}$ is the conformal map taking 0 to $-i$, ∞ to i , and 0 to i , then $h \circ \psi - \chi \arg \psi'$ is a GFF on \mathbf{H} with boundary data $-\lambda$ on $(-\infty, 0]$ and λ on $[0, \infty)$. By [MS16a, Theorem 1.1] the zero-angle flow line η of h started from $-i$ is a chordal SLE $_\kappa$ from $-i$ to i in \mathbf{D} .³

Fix $\Delta > \tilde{\Delta} > 0$, $q \in (-1/2, \infty)$, $\dot{\Delta}, c, \theta, \delta_0, r, \dot{r}, p_L > 0$, $a \in (0, 1/4)$, and $\mu, \mu_L, \mu_F \in \mathcal{M}$. Assume that the parameters a and μ are chosen in such a way that the conclusion of Proposition 6.1 holds for $x = -i$ and $y = i$. Also fix sequences $\beta_j \rightarrow \infty$ and $u_j \rightarrow 0$, which we will choose in Section 6.3 below.

For each $z \in \mathbf{D}$ and $j \in \mathbf{N}$, we will inductively define the following objects.

- Events $L_{z,j}, \tilde{E}_{z,j}, F_{z,j}, E_{z,j}$.
- Points $x_{z,j}, y_{z,j}, x_{z,j}^*, y_{z,j}^*, x_{z,1}^F, y_{z,1}^F, b_{z,j}, \bar{b}_{z,j}$.
- Conformal maps $\phi_{z,j}, p_{z,j}, \hat{p}_{z,j}, \psi_{z,j}, \psi_{z,j}^F$.
- Random times $\sigma_{z,j}, \bar{\sigma}_{z,j}, \tau_{z,j}, \bar{\tau}_{z,j}, \tau_{z,1}^*, \bar{\tau}_{z,1}^*, T_{z,j}, \bar{T}_{z,j}, T_{z,j}^*, \bar{T}_{z,j}^*, t_{z,j}^\pm$, and $\tilde{t}_{z,j}^\pm$.
- Curves $\eta_{z,j}, \tilde{\eta}_{z,j}, \eta_{z,j}^\pm$, and $\tilde{\eta}_{z,j}^\pm$.
- Domains $D_{z,j}$ and $\hat{D}_{z,j}$.
- σ -algebras $\mathcal{F}_{z,j}$.

³In the case $\kappa = 4$, we replace flow lines of h with a given angle by level lines of h at a given level (see [SS09, SS13, WW14]). Everything that follows works identically with this replacement. In fact, since (in contrast to the situation for flow lines) the time reversal of a level line is also a level line [WW14, Theorem 1.1.5], some of the proofs are easier for $\kappa = 4$.

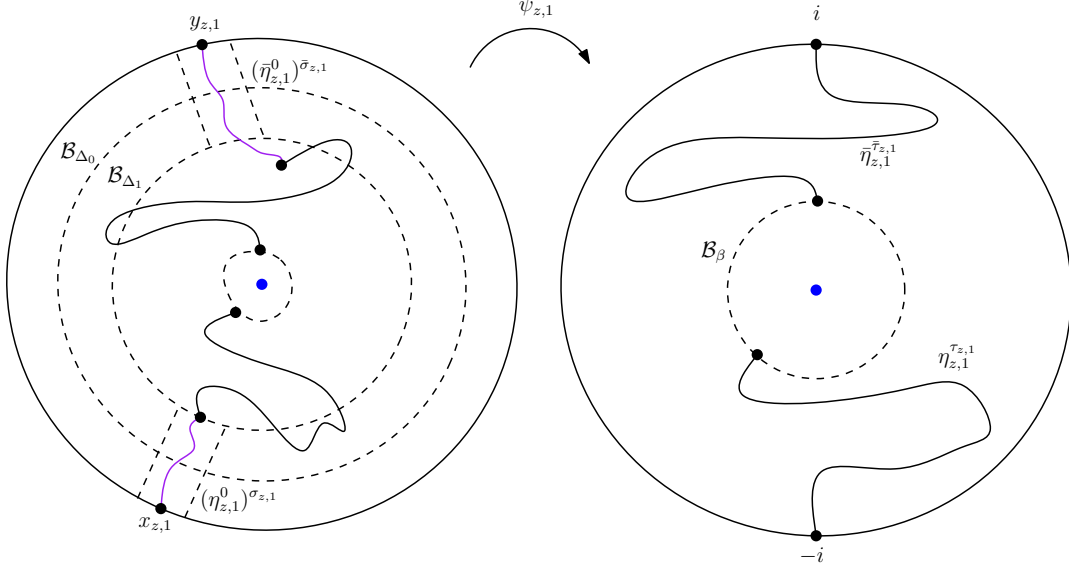


Figure 6.2: An illustration of the parts of the curve $\eta_{z,1}^0$ (left) and $\eta_{z,1}$ (right) associated with the events $L_{z,1}$ and $\tilde{E}_{z,1}$. For clarity, the disks here are shown larger than they actually are in practice. The same is true in the other figures in this section.

First consider the case $j = 1$. Let $f_{z,1}$ be the conformal automorphism of \mathbf{D} satisfying $f_{z,1}(z) = 0$ and $f'_{z,1}(0) > 0$. Let $x_{z,1} = x_{z,1}^* = f_{z,1}(-i)$ and $y_{z,1} = y_{z,1}^* = f_{z,1}(i)$. Let $\eta_{z,1}^0 = f_{z,1}(\eta)$.

Let $\sigma_{z,1}$ be the first time $\eta_{z,1}^0$ hits \mathcal{B}_{Δ} . For $\alpha \geq 0$ let \bar{t}_{α} be the first time $\bar{\eta}_{z,1}$ hits \mathcal{B}_{α} . Let $\psi_{z,1}^{\alpha}$ be the conformal map from the connected component of $\mathbf{D} \setminus ((\eta_{z,1}^0)^{\sigma_{z,1}} \cup (\bar{\eta}_{z,1}^0)^{\bar{t}_{\alpha}})$ containing the origin to \mathbf{D} which fixes 0 and takes $\eta_{z,1}^0(\sigma_{z,1})$ to $-i$.⁴

Let $\bar{\sigma}_{z,1} = \bar{t}_{\alpha}$ be the smallest $\alpha \geq \Delta$ such that $\psi_{z,1}^{\alpha}(\bar{\eta}_{z,1}^0(\bar{t}_{\alpha})) = i$, provided such an α exists, and $\bar{\sigma}_{z,1} = \infty$ otherwise. Let $\psi_{z,1} = \psi_{z,1}^{\alpha}$ for this α if such an α exists and let $\psi_{z,1} = \text{Id}$ otherwise. Let $L_{z,1}$ be the event that the following occurs.

1. $(\eta_{z,1}^0)^{\sigma_{z,1}}$ (resp. the part of $(\bar{\eta}_{z,1}^0)^{\bar{\sigma}_{z,1}}$ traced before it hits \mathcal{B}_{Δ}) is contained in the $e^{-2\Delta}$ -neighborhood of the segment $[x_{z,1}, 0]$ (resp. $[y_{z,1}, 0]$).
2. $\eta_{z,1}^0$ and $\bar{\eta}_{z,1}^0$ do not leave \mathcal{B}_{Δ_0} after hitting $\mathcal{B}_{\Delta/2}$.
3. $\bar{\sigma}_{z,1} \leq \bar{t}_{\Delta+\log 2}$ and $\bar{\eta}_{z,1}^0([t_{\Delta}, \bar{\sigma}_{z,1}]) \subset \mathcal{B}_{\Delta/2}$.
4. $\psi_{z,1}^{-1}$ maps $B_{1-\mu(\delta_0)}(0) \cup B_{\delta_0}(-i) \cup B_{\delta_0}(i)$ into $\mathcal{B}_{\tilde{\Delta}}$.
5. $e^{-\tilde{\Delta}+\tilde{r}/4} \leq |\psi'_{z,1}(0)| \leq e^{-\tilde{\Delta}+\tilde{r}/2}$.
6. $\mathcal{G}_{[x_{z,1}^*, y_{z,1}^*]}(\psi_{z,1}, \mu_L)$ occurs (Definition 2.5).
7. The conditional probability given $(\eta_{z,j}^0)^{\sigma_{z,j}} \cup (\bar{\eta}_{z,j}^0)^{\bar{\sigma}_{z,j}}$ that the part of $\eta_{z,1}^0$ lying between $\eta_{z,1}^0(\sigma_{z,1})$ and $\bar{\eta}_{z,1}^0(\bar{\sigma}_{z,1})$ never exits $\mathcal{B}_{\tilde{\Delta}}$ is at least p_L .

Let $\eta_{z,1}$ be the image under $\psi_{z,1}$ of the part of $\eta_{z,1}^0$ lying between $\eta_{z,1}^0(\sigma_{z,1})$ and $\bar{\eta}_{z,1}^0(\bar{\sigma}_{z,1})$. Let $\mathcal{F}_{z,1}^0$ be the σ -algebra generated by $\eta_{z,1}^0|_{[0, \sigma_{z,1}]}$ and $\bar{\eta}_{z,1}^0|_{[0, \bar{\sigma}_{z,1}]}$.

⁴This connected component is all of $\mathbf{D} \setminus ((\eta_{z,1}^0)^{\sigma_{z,1}} \cup (\bar{\eta}_{z,1}^0)^{\bar{t}_{\alpha}})$ in the case $j = 1$, but may not be in the case $j \geq 2$ since then $\eta_{z,j}^0$ may hit $\partial\mathbf{D}$.

Remark 6.6. The reason for introducing $\eta_{z,1}$ instead of working directly with $\eta_{z,1}^0$ is that we need the laws of the curves at each stage in our construction to be s.m.a.c. (Definition C.1). Otherwise we will end up with additional proportionality constants in our probability estimates. When we iterate these estimates several times, the proportionality constants will produce an exponential factor which will have a significant impact on our final estimates. The reason for condition 4 in the definition of $L_{z,1}$ is that it implies $\psi_{z,1}^{-1}(\eta_{z,1}) \subset \mathcal{B}_{\tilde{\Delta}}$ provided condition 4 in the definition of $E_{\beta_1}^{q;u_1}(\eta_{z,1}, c, a, \mu)$ holds.

Remark 6.7. By [MW14, Lemma 2.3], reversibility, and the Markov property, condition 7 in the definition of $L_{z,1}$ (for sufficiently small p_L) follows from the other conditions in the case of an ordinary SLE_κ . However, when we define the events $L_{z,j}$ for $j \geq 2$, the curve $\eta_{z,j}^0$ will be an $\text{SLE}_\kappa(\rho^L; \rho^R)$ for certain $\rho^L, \rho^R \in (-2, 0)$, in which case condition 7 (with j in place of 1) is non-trivial. The reason for introducing condition 7 is as follows. By Lemma C.4, we can estimate the law of the remainder of the middle part of $\eta_{z,j}^0$ given $(\eta_{z,1}^0)^{\sigma_{z,1}} \cup (\bar{\eta}_{z,1}^0)^{\bar{\sigma}_{z,1}}$ conditioned on the event that this curve does not leave \mathcal{B}_Δ . However, we want to estimate the law *restricted* to the event that the curve does not leave \mathcal{B}_Δ . For this, we need the probability that it leaves \mathcal{B}_Δ to be $\asymp 1$.

Lemma 6.8. *For any $\tilde{\Delta} > 0$, $\delta_0 > 0$, and $\mu \in \mathcal{M}$, and $\dot{r} > 0$, we can find $\Delta > 0$, and $\dot{\Delta} > 0$ depending only on Δ , $\mu_L \in \mathcal{M}$ depending only on μ , and $p_L > 0$ depending only on the other parameters in the definition of $L_{z,1}$, all depending uniformly on z in compacts, such that $\mathbf{P}(L_{z,1}) \geq 1$, with the implicit constant uniform for z in compacts.*

Proof. This follows from [MW14, Lemma 2.3]. Note that we apply the Koebe growth theorem to $\psi_{z,1}^{-1}$ in order to make $\tilde{\Delta}$ as large as we like. \square

Let

$$\tilde{E}_{z,1} = L_{z,1} \cap E_{\beta_1}^{q;u_1}(\eta_{z,1}, c, a, \mu),$$

with the latter event as in Section 6.1. Let $\tau_{z,1}$, and $\bar{\tau}_{z,1}$ be the stopping times τ_{β_1} and $\bar{\tau}_{\beta_1}$ from Section 6.1. Let $\phi_{z,1} : \mathbf{D} \setminus (\eta_{z,1}^{\tau_{z,1}} \cup \bar{\eta}_{z,1}^{\bar{\tau}_{z,1}}) \rightarrow \mathbf{D}$ be the conformal map ϕ_{β_1} from Section 6.1. Let $T_{z,1}$ be the time for η corresponding to $\tau_{z,1}$.

Let $\hat{\eta}_{z,1}^+$ and $\hat{\eta}_{z,1}^-$ be the flow lines of h started from $\eta(T_{z,1})$ with angles θ and $-\theta$, respectively.⁵ Note that the flow line with a negative sign has positive angle and vice versa. This is because a flow line with a negative angle a.s. stays to the right of η , and a flow line with a positive angle a.s. stays to the left of η . See [MS16a, Theorem 1.5]. Let $\eta_{z,1}^\pm = (\psi_{z,1} \circ f_{z,1})(\hat{\eta}_{z,1}^\pm)$.

By examining the boundary data of the field h along η and applying [MS16a, Theorems 1.1 and 2.4], we find that the conditional law of $\hat{\eta}_{z,1}^+$ (resp. $\hat{\eta}_{z,1}^-$) given η is that of a $\text{SLE}_\kappa(\rho^0; \rho^1)$ processes from $\eta(T_{z,1})$ to i in the right (resp. left) connected component of $\mathbf{D} \setminus \eta$, where

$$\rho^0 = -\frac{\theta\chi}{\lambda}, \quad \rho^1 = \frac{\theta\chi}{\lambda} - 2 \quad (6.12)$$

and the force points are located immediately to the left and right of $\eta(T_{z,1})$.

By [MS16a, Remark 5.3], $\hat{\eta}_{z,1}^\pm$ a.s. fail to hit $\eta^{T_{z,1}} \cup \partial\mathbf{D}$ provided $-\theta\chi/\lambda \geq \kappa/2 - 2$. Furthermore, $\eta_{z,1}^\pm$ a.s. intersect (but do not cross) each other provided $2\theta\chi/\lambda - 2 < \kappa/2 - 2$. Since $\kappa \leq 4$ we can choose a small $\theta > 0$ depending only on κ in such a way that $\eta_{z,1}^\pm$ a.s. do not intersect $\partial\mathbf{D}$ but do a.s. intersect each other. We henceforth assume that θ has been chosen in this manner.

Let $\hat{D}_{z,1}$ be the connected component of $\mathbf{D} \setminus (\hat{\eta}_{z,1}^+ \cup \hat{\eta}_{z,1}^-)$ which contains z . Let $D_{z,1}$ be the connected component of $\mathbf{D} \setminus (\eta_{z,1}^+ \cup \eta_{z,1}^-)$ which contains the origin. Let $p_{z,1} : D_{z,1} \rightarrow \mathbf{D}$ be the conformal map with $p_{z,1}(0) = 0$ and $p'_{z,1}(0) > 0$. Let $\hat{p}_{z,1} := p_{z,1} \circ \psi_{z,1} \circ f_{z,1}$, so that $\hat{p}_{z,1} : \hat{D}_{z,1} \rightarrow \mathbf{D}$ and takes $\hat{p}_{z,1}(0) = 0$. See Figure 6.3 for an illustration of the event $\tilde{E}_{z,1}$ and the flow lines $\hat{\eta}_{z,1}^\pm$.

Let $t_{z,1}^+$ be the first time that $\eta_{z,1}^+$ hits $\eta_{z,1}^-$ after the first time it exits the disk of radius $e^{-\beta_1 - \dot{r}}$ centered at $\eta_{z,1}(\tau_{z,1})$. Let $t_{z,1}^-$ be the time t such that $\eta_{z,1}^-(t) = \eta_{z,1}^+(t_{z,1}^+)$. Let $\bar{b}_{z,1} = \eta_{z,1}^\pm(t_{z,1}^\pm)$ and let $b_{z,1}$ be

⁵Since $T_{z,1}$ is not a stopping time for η , we define the flow lines $\hat{\eta}_{z,1}^\pm$ by taking the limit of the flow lines of angles $\mp\theta$ of h started from $\eta(t)$ as t increases to $T_{z,1}$ along rational times. The limiting object can equivalently be defined as the left or right outer boundary of a certain counterflow line of h started from $-i$ and stopped at the first time it hits $\eta(T_{z,1})$ (c.f. [MS16a, Section 5]), so in particular is a simple curve.

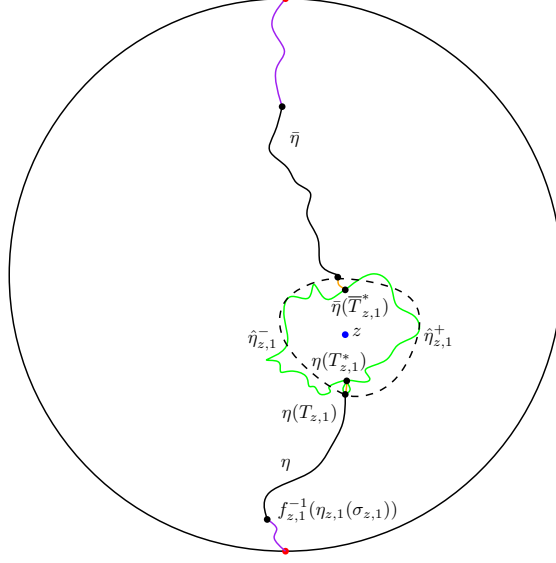


Figure 6.3: An illustration of the curve η grown up to its hitting time of a quasi-disk centered at z on the event $E_{z,1}$, together with the flow lines $\hat{\eta}_{z,1}^\pm$ (shown in green) used in the definition of the event $F_{z,1}$. The part of the curve η associated with the event $L_{z,1}$ is shown in purple. The domain $\hat{D}_{z,1}$ is the complementary connected component of the green flow lines which contains z . Also shown are the segments of η involved in the last part of the event $E_{z,1}$ (in orange) and the stopping times $T_{z,1}^*$ and $\bar{T}_{z,1}^*$ defined just after the definition of $E_{z,1}$.

the last intersection point of $\eta_{z,1}^\pm$ before $\bar{b}_{z,1}$. Also let $\tilde{t}_{z,1}^\pm$ be the first exit times of $\eta_{z,1}^\pm$ from the annulus $\mathcal{B}_{\beta_1 - \log 2} \setminus \mathcal{B}_{\beta_1 + \log 2}$. Let $F_{z,1}$ be the event that the following occurs.

1. $t_{z,1}^+ \leq \tilde{t}_{z,1}^+$ and $t_{z,1}^- \leq \tilde{t}_{z,1}^-$.
2. $|b_{z,1}| \leq e^{-\beta_1 - \dot{r}}$ and $\bar{b}_{z,1} \notin \bar{\eta}_{z,1}^{\bar{\tau}_{z,1}^*}$.
3. Let $\psi_{z,1}^F : \mathbf{D} \setminus (\eta_{z,1}^{\tau_{z,1}^*} \cup \bar{\eta}_{z,1}^{\bar{\tau}_{z,1}^*})$ be the conformal map with $\psi_{z,1}^F(0) = 0$ and $(\psi_{z,1}^F)'(0) > 0$. Let $x_{z,1}^F = \psi_{z,1}^F(\eta_{z,1}(\tau_{z,1}))$ and $y_{z,1}^F = \psi_{z,1}^F(\bar{\eta}_{z,1}(\bar{\tau}_{z,1}))$. Then $|\psi_{z,1}^F(b_{z,1}) - x_{z,1}^F|$ and $|\psi_{z,1}^F(\bar{b}_{z,1}) - y_{z,1}^F|$ are each at most r .
4. Each point of $\psi_{z,1}^F((\eta_{z,1}^+)^{t_{z,1}^+})$ (resp. $\psi_{z,1}^F((\bar{\eta}_{z,1}^+)^{\bar{t}_{z,1}^+})$) lies within distance r of $[x_{z,1}^F, y_{z,1}^F] \cap \partial \mathbf{D}$ (resp. $[y_{z,1}^F, x_{z,1}^F] \cap \partial \mathbf{D}$).
5. $\mathcal{G}'(\psi_{z,1}^F((\eta_{z,1}^+)^{t_{z,1}^+} \cup (\eta_{z,1}^-)^{t_{z,1}^-}), \mu_F)$ occurs.

See Figure 6.4 for an illustration of the event $F_{z,1}$. It follows from condition 3 that $D_{z,1}$ is the “pocket” formed by $\eta_{z,1}^\pm$ between their hitting times of $b_{z,1}$ and $\bar{b}_{z,1}$ on the event $F_{z,1}$.

Remark 6.9. Our reason for introducing the auxiliary flow lines $\eta_{z,1}^\pm$ is as follows. The conditional law of the part of η which lies inside $\hat{D}_{z,1}$ is conditionally independent of the part of η which lies outside $\hat{D}_{z,1}$ given the flow lines $\eta_{z,1}^\pm$ (see Lemma 6.21 below). When applied at various scales, this fact will eventually allow us to get the needed “near independence” of the events $E_{z,j}$ and $E_{w,j}$ for $z \neq w$.

Let $\tau_{z,1}^*$ (resp. $\bar{\tau}_{z,1}^*$) be the time $\eta_{z,1}$ (resp. $\bar{\eta}_{z,1}$) hits $b_{z,1}$ (resp. $\bar{b}_{z,1}$). Note that these times are a.s. finite since $\eta_{z,1}$ a.s. lies between $\eta_{z,1}^\pm$. Let $E_{z,1}$ be the event that the following occurs.

1. $\tilde{E}_{z,1} \cap F_{z,1}$ occurs.

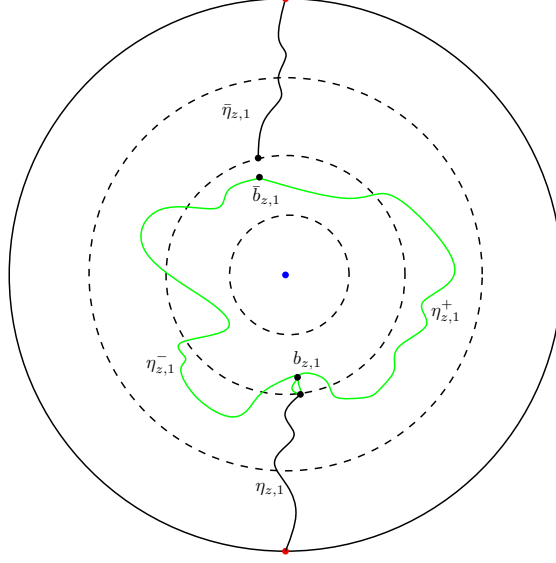


Figure 6.4: An illustration of the event $F_{z,1}$ that $\tilde{E}_{z,1}$ occurs and the flow lines $\eta_{z,1}^\pm$ started at $\eta_{z,1}(\tau_{z,1})$ behave in the manner we would like. The domain $D_{z,1}$ is that which lies between the parts of $\eta_{z,1}^\pm$ traced between their hitting times of $b_{z,1}$ and $\bar{b}_{z,1}$.

2. $\psi_{z,1}^F(\eta_{z,1}([\tau_{z,1}, \tau_{z,1}^*]))$ (resp. $\psi_{z,1}^F(\bar{\eta}_{z,1}([\bar{\tau}_{z,1}, \bar{\tau}_{z,1}^*]))$) is contained in the disk of radius $2r$ centered at $x_{z,1}^F$ (resp. $y_{z,1}^F$) (notation as in condition 3 in the definition of $F_{z,1}$).

Remark 6.10. By [MS16a, Theorem 1.5] $\eta_{z,1}$ cannot cross $\eta_{z,1}^\pm$. By combining this with condition 4 in the definition of $L_{z,1}$, condition 4 in the definition of $E_{\beta_1}^{q;u_1}(\cdot)$, and condition 2 in the definition of $E_{z,1}$, it follows that the whole of $\psi_{z,1}^{-1}(\eta_{z,1})$ (i.e. the part of $\eta_{z,1}^0$ between $\eta_{z,1}^0(\sigma_{z,1})$ and $\bar{\eta}_{z,1}^0(\bar{\sigma}_{z,1})$) is contained in $\mathcal{B}_{\tilde{\Delta}}$ on the event $E_{z,1}$.

Let $T_{z,1}^*$ and $\bar{T}_{z,1}^*$, resp., be the times for η and $\bar{\eta}$, resp., corresponding to $\tau_{z,1}^*$ and $\bar{\tau}_{z,1}^*$. Let $\mathcal{F}_{z,1}$ be the σ -algebra generated by $\eta|_{[0, T_{z,1}^*]}$, $\bar{\eta}|_{[0, \bar{T}_{z,1}^*]}$, and $\tilde{\eta}_{z,1}^\pm|_{[0, t_{z,1}^\pm]}$.

Lemma 6.11. *Given $r > 0$ we can choose μ_F and \dot{r} independently of β_1 and u_1 and uniform for z in compacts in such a way that $\mathbf{P}(E_{z,1} | \tilde{E}_{z,1}) \asymp 1$ with the implicit constants depending on the other parameters but not on β_1 or u_1 , and uniform for z in compacts.*

Proof. Let $\eta_{z,1}^F$ be the image under $\psi_{z,1}^F$ of the part of $\eta_{z,1}$ between $\eta_{z,1}(\tau_{z,1})$ and $\bar{\eta}_{z,1}(\bar{\tau}_{z,1})$. Note that the distance between the endpoints $x_{z,1}^F$ and $y_{z,1}^F$ of $\eta_{z,1}^F$ is uniformly positive on $\tilde{E}_{z,1}$ by condition 2 in the definition of $E_{\beta_1}^{q;u_1}(\cdot)$. Let $\tilde{r} \in (0, r)$ and let U be the \tilde{r} -neighborhood of the line segment from $x_{z,1}^F$ to $y_{z,1}^F$. Let $\mu_F' \in \mathcal{M}$. Let $S_{z,1}$ be the event that $\eta_{z,1}^F \subset U$, $\mathcal{G}'(\eta_{z,1}^F, \mu_F')$ occurs, and the time reversal of $\eta_{z,1}^F$ does not enter $B_r(y_{z,1}^F)$ after leaving $B_{2r}(y_{z,1}^F)$.

By the Markov property and reversibility of SLE, the conditional law of $\eta_{z,1}^F$ given a realization of $\mathcal{F}_{z,1}^0$ for which $L_{z,1}$ occurs and a realization of $\eta_{z,1}^{\tau_{z,1}} \cup \bar{\eta}_{z,1}^{\bar{\tau}_{z,1}}$ is that of a chordal SLE_κ from $x_{z,1}^F$ to $y_{z,1}^F$ in \mathbf{D} . By [MW14, Lemma 2.3], we thus have $\mathbf{P}(S_{z,1} | \tilde{E}_{z,1}) \gtrsim 1$ provided μ_F' is chosen sufficiently small, independently of β_1 , u_1 , and z .

The conditional law of $\psi_{z,1}^F(\eta_{z,1}^+)$ given $\eta_{z,1}^0$ is that of a $\text{SLE}_\kappa(\rho^0; \rho^1)$ process in the right connected component of $\mathbf{D} \setminus \eta_{z,1}^F$ from $x_{z,1}^F$ to $\psi_{z,1}^F(i^-)$ with force points located on either side of $x_{z,1}^F$, where ρ^0 and ρ^1 are as in (6.12). Our choice of θ implies that such a process a.s. does not hit $[x_{z,1}^F, \psi_{z,1}^F(i^-)] \partial \mathbf{D}$ (see [MS16a, Section 4] or [Dub09a, Lemma 15]). Condition 2 in the definition of $E_{\beta_1}^{q;u_1}(\cdot)$ implies that $\psi_{z,1}^F(i^-)$ lies at uniformly positive distance from $x_{z,1}^F$ and $y_{z,1}^F$ on $\tilde{E}_{z,1}$. Similar statements hold with $-$ in

place of $+$ and “left” in place of “right”. By [MW14, Lemma 2.5] (and a straightforward complex analysis argument to make a suitable choice of \dot{r}), we have $\mathbf{P}(F_{z,1} | \tilde{E}_{z,1} \cap S_{z,1}) \succeq 1$ provided \tilde{r} , μ_F , and \dot{r} are chosen sufficiently small, independently of β_1 , u_1 , and z .⁶ If $F_{z,1} \cap \tilde{E}_{z,1} \cap S_{z,1}$ occurs, then so does $E_{z,1}$. We conclude by observing that

$$\mathbf{P}(E_{z,1} | \tilde{E}_{z,1}) \geq \mathbf{P}(F_{z,1} \cap S_{z,1} | \tilde{E}_{z,1}) = \mathbf{P}(F_{z,1} | \tilde{E}_{z,1} \cap S_{z,1}) \mathbf{P}(S_{z,1} | \tilde{E}_{z,1}).$$

□

By combining Proposition 6.1, Lemma 6.8, and Lemma 6.11, we infer the following one-point estimate for the event $E_{z,1}$.

Lemma 6.12. *Provided β_1 is chosen sufficiently large (how large depends on u_1 and the other parameters but is uniform for z in compacts) and the other parameters are chosen appropriately, independently of β_1 and u_1 and uniformly for z in compacts, we have*

$$e^{-\beta_1(\gamma^*(q) + \gamma_0^*(q)u_1)} \preceq \mathbf{P}(E_{z,1}) \preceq e^{-\beta_1(\gamma^*(q) - \gamma_0^*(q)u_1)}$$

with the implicit constant depending on the other parameters (including u_1) but not on β_1 , and uniform for z in compacts.

Now suppose $j \geq 2$ and the objects have been defined for all positive integers $l \leq j-1$.

Let $\eta_{z,j}^0$ be the image under $p_{z,j-1}$ of the part of $\eta_{z,j-1}$ which lies in $D_{z,j-1}$. Let $x_{z,j}$ and $y_{z,j}$ be its initial and terminal points. Define the map $\psi_{z,j}$, the times $\sigma_{z,j}$ and $\bar{\sigma}_{z,j}$, and the curve $\eta_{z,j}$ in the same manner as in the case $j=1$, but with all of the 1’s replaced by j ’s (the auxiliary parameters remain unchanged).

Let $[x_{z,j}^*, y_{z,j}^*]_{\partial \mathbf{D}}$ be the largest sub-arc of $[x_{z,j}, y_{z,j}]_{\partial \mathbf{D}}$ not disconnected from the origin by $(\eta_{z,j}^0)^{\sigma_{z,j}} \cup (\bar{\eta}_{z,j}^0)^{\bar{\sigma}_{z,j}}$ (this arc need not be equal to $[x_{z,j}, y_{z,j}]_{\partial \mathbf{D}}$ since $\eta_{z,j}^0$ can intersect $\partial \mathbf{D}$). Fix a constant $p_L > 0$. Define the event $L_{z,j}$ in exactly the same manner as in the case $j=1$ but with all 1’s replaced by j ’s.

Define the event $\bar{E}_{z,j}$, the times $\tau_{z,j}$ and $\bar{\tau}_{z,j}$, and the map $\phi_{z,j}$ in exactly the same manner as the corresponding objects for $j=1$, but with all the 1’s replaced by j ’s. Let $T_{z,j}$ and $\bar{T}_{z,j}$ be the times for η and $\bar{\eta}$ corresponding to $\tau_{z,j}$ and $\bar{\tau}_{z,j}$.

Let $\hat{\eta}_{z,j}^\pm$ be the flow lines of η started from $\eta(T_{z,j})$ with angles $\mp\theta$. Let $\hat{D}_{z,j}$ be the connected component of $\mathbf{D} \setminus (\hat{\eta}_{z,j}^+ \cup \hat{\eta}_{z,j}^-)$ containing z . Let $\eta_{z,j}^\pm$ be the images of $\hat{\eta}_{z,j}^\pm$ under the map $\psi_{z,j} \circ \hat{p}_{z,j-1}$. Define the domain $D_{z,j}$, the event $F_{z,j}$, the map $p_{z,j}$, and the times $t_{z,j}^\pm$ and $\bar{t}_{z,j}^\pm$ in exactly the same manner as in the $j=1$ case except with all of the 1’s replaced by j ’s. Let $\hat{p}_{z,j} = p_{z,j} \circ \psi_{z,j} \circ \hat{p}_{z,j-1}$ so that $\hat{p}_{z,j} : \hat{D}_{z,j} \rightarrow \mathbf{D}$ and $\hat{p}_{z,j}(0) = 0$.

Also define the event $E_{z,j}$ and the times $\tau_{z,j}^*$ and $\bar{\tau}_{z,j}^*$ in exactly the same manner as in the $j=1$ case except with all of the 1’s replaced by j ’s. Let $T_{z,j}^*$ and $\bar{T}_{z,j}^*$ be the times for η and $\bar{\eta}$ corresponding to $\tau_{z,j}^*$ and $\bar{\tau}_{z,j}^*$. Let $\mathcal{F}_{z,j}$ be the σ -algebra generated by $\eta|_{[0, T_{z,j}^*]}, \bar{\eta}|_{[0, \bar{T}_{z,j}^*]}$, and $\hat{\eta}_{z,l}^\pm|_{[0, t_{z,l}^\pm]}$ for $l \leq j$.

6.3 k-perfect points

Fix sequences $\beta_j \rightarrow \infty$ and $u_j \rightarrow 0$ as above. Also fix $d \in (0, 1)$. The estimate of Lemma 6.12 tells us that if β_j is chosen sufficiently large (depending on u_j but uniform for $z \in B_d(0)$) then we have

$$C_{u_j}^{-1} e^{-\beta_j(\gamma^*(q) + \gamma_0^*(q)u_j)} \leq \mathbf{P}(E_{z,j}) \leq C_{u_j} e^{-\beta_j(\gamma^*(q) - \gamma_0^*(q)u_j)}, \quad (6.13)$$

where for $u > 0$, C_u is a constant which is allowed to depend on u and on the other parameters in the preceding subsection but not on any of the β_j ’s, and is uniform for $z \in B_d(0)$.

⁶To get that the flow lines $\eta_{z,1}^\pm$ intersect one another where we want them to with uniformly positive probability, we can further condition on a second pair of flow lines $\hat{\eta}_{z,1}^\pm$ with the same angles as $\eta_{z,1}^\pm$, started at a point near where we want the intersection to occur. We then apply [MW14, Lemma 2.5] to the conditional law of $\eta_{z,1}^\pm$ given $\hat{\eta}_{z,1}^\pm$ and η , and observe that $\eta_{z,1}^\pm$ merge with $\hat{\eta}_{z,1}^\pm$ upon intersecting [MS16a, Theorem 1.5]; and that $\hat{\eta}_{z,1}^\pm$ a.s. intersect one another at points arbitrarily close to their starting points. See [MW14] for several examples of similar arguments.

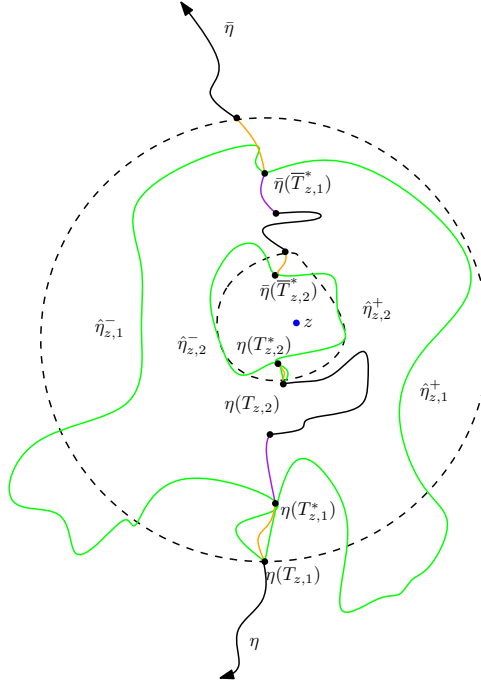


Figure 6.5: An illustration of the curve η and the auxiliary flow lines $\hat{\eta}_{z,1}^\pm$ and $\hat{\eta}_{z,2}^\pm$ on the event $E_{z,1} \cap E_{z,2}$. Note that only a neighborhood of z is shown, not the whole disk. The parts of η involved with the last parts of the definitions of $E_{z,1}$ and $E_{z,2}$ are shown in orange. The parts of η involved with the event $L_{z,2}$ are shown in purple. The domain $\hat{D}_{z,1}$ (resp. $\hat{D}_{z,2}$) is the region enclosed by the outer (resp. inner) green curves.

Remark 6.13. The reason we allow β and u to vary here is that we eventually want to get a lower bound for the Hausdorff dimension of the sets $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$. If we fixed u , we would instead get the Hausdorff dimension of the sets where the limits in the definitions of $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$ are between $s - u$ and $s + u$. In order to allow u to vary, we also need to allow β to vary, for otherwise the constants C_u in (6.13) would be larger than e^β when u is very small. The idea in Lemma 6.14 below is to let $u_j \rightarrow 0$ and $\beta_j \rightarrow \infty$ slowly enough that our estimates are not much different than they would be with fixed β and u .

Lemma 6.14. *Given d and the parameters from Section 6.2, one can choose sequences $(\beta_j)_{j \in \mathbf{N}}$ and $(u_j)_{j \in \mathbf{N}}$ such that the following is true.*

1. β_j increases to ∞ and u_j decreases to 0 as $j \rightarrow \infty$.
2. For each $j \in \mathbf{N}$, $\beta_{j+1} \leq \beta_j + o_j(1)$.
3. For each fixed $k \in \mathbf{N}$, $\beta_{jk} \leq \bar{\beta}_j o_j(1)$.
4. The estimate (6.13) holds for each $j \in \mathbf{N}$.
5. For each $j \in \mathbf{N}$, $C_{u_j} \leq e^{\beta_j u_j \gamma_0^*(q)}$.
6. $\beta_j u_j \rightarrow \infty$ as $j \rightarrow \infty$.

Proof. Fix $u_0 \in (0, 1)$. Choose β_0 much larger than $\Delta \vee \gamma_0^*(q)^{-1} \log C_{u_0}$ and large enough that (6.13) holds with β_0 in place of β_j and u_0 in place of u_j . Set $\beta_j = \log j + \beta_0$. It is clear that $\beta_j \rightarrow \infty$ as $j \rightarrow \infty$ and assertions 2 and 3 hold. We now inductively choose $(u_j)_{j \in \mathbf{N}}$.

Start with a sequence $(u_l^*)_{l \in \mathbf{N}} \subset (0, u_0)$ which decreases to 0. Let j_1 be the least positive integer j such that (6.13) holds with $u_j = u_1^*$, $C_{u_1^*} \leq e^{\beta_{j_1} u_1^* \gamma_0^*(q)}$, and $\beta_{j_1} u_1^* \geq 1$. Such a j exists since $\beta_j \rightarrow \infty$ as $j \rightarrow \infty$.

Set $u_j = u_0$ for $j \in \{1, \dots, j_1\}$. Inductively, suppose $l \geq 1$ and j_1, \dots, j_{l-1} and u_j for $j \leq j_{l-1}$ have been defined. Let j_l be the least integer $j \geq j_{l-1} + 1$ such that (6.13) holds with $u_j = u_l^*$, $C_{u_l^*} \leq e^{\beta_j u_l^* \gamma_0^*(q)}$, and $\beta_j u_l^* \geq l$. Let $u_j = u_{l-1}^*$ for $j \in \{j_{l-1} + 1, \dots, j_l\}$. It is clear that conditions 4, 5, and 6 hold for this choice of (u_j) . \square

We henceforth assume the sequences (β_j) and (u_j) are chosen as in Lemma 6.14. We also let

$$\bar{\beta}_{m_1, m_2} = \sum_{j=m_1+1}^{m_2} \beta_j, \quad \bar{u}_{m_1, m_2} = \sum_{j=m_1+1}^{m_2} \beta_j u_j, \quad \bar{\beta}_m = \bar{\beta}_{0, m}, \quad \bar{u}_m = \bar{u}_{0, m}. \quad (6.14)$$

For $m \in \mathbf{N}$, define the events $E_{z, j}^m$ in the same manner as the events $E_{z, j}$ in the preceding subsection but with the sequences (β_j) and (u_j) replaced by (β_{j+m}) and (u_{j+m}) . For $n \geq k \geq 0$, define

$$E_{k, n}^m(z) = \bigcap_{j=k+1}^n E_{z, j}^m. \quad (6.15)$$

Also let

$$E_{k, n}(z) := E_{k, n}^0(z), \quad E_n(z) := E_{0, n}(z).$$

The set of n -perfect points is defined by

$$\mathcal{P}_n := \{z \in \mathbf{D} : E_n(z) \text{ occurs}\} \quad (6.16)$$

6.4 Analytic properties

In this subsection we study some analytic properties of the events of Sections 6.2 and 6.3. The results of this subsection are needed to analyze the correlation structure of our events in the next subsection and to show that the perfect points are in fact contained in the sets whose Hausdorff dimension we want to compute in Section 7. The main result of this subsection is the following.

Lemma 6.15. *Assume we are in the setting of Sections 6.2 and 6.3. For $n \in \mathbf{N}$ let $\Phi_{z, n}$ be the conformal map from $\mathbf{D} \setminus (\eta^{T_{z, n}^*} \cup \bar{\eta}^{\bar{T}_{z, n}^*})$ to \mathbf{D} which takes $-i^+$ to $-i$, i^- to i , and 1 to 1 . If we choose the parameter r sufficiently small and β_1 (and hence every β_j) sufficiently large, in a manner which does not depend on (u_j) and is uniform for $z \in B_d(0)$, then the following holds a.s. on $E_n(z)$, with all implicit constants deterministic and independent of n and of $z \in B_d(0)$.*

1. We have

$$e^{-\bar{\beta}_n q - 2\bar{u}_n} \leq |\Phi'_{z, n}(z)| \leq e^{-\bar{\beta}_n q + 2\bar{u}_n}.$$

2. There is a constant $\lambda > 0$, independent of n and $z \in B_d(0)$, such that

$$e^{-\bar{\beta}_n - \lambda n} \leq \text{dist} \left(z, \eta^{T_{z, n}^*} \cup \bar{\eta}^{\bar{T}_{z, n}^*} \right) \leq e^{-\bar{\beta}_n + \lambda n}.$$

3. We have

$$|\eta(T_{z, n}^*) - z| \asymp |\bar{\eta}(\bar{T}_{z, n}^*) - z| \asymp \text{dist} \left(z, \eta^{T_{z, n}^*} \cup \bar{\eta}^{\bar{T}_{z, n}^*} \right)$$

4. We have

$$e^{-\bar{\beta}_n - \lambda n} \leq \text{dist}(z, \partial \hat{D}_{z, n}) \leq \text{diam} \hat{D}_{z, n} \leq e^{-\bar{\beta}_n + \lambda n}.$$

To prove Lemma 6.15 we will need to compare the derivatives of several different maps. To this end, we will define the following objects.

- Conformal maps $\phi_{z, j}^0$, $\tilde{\phi}_{z, j}$, $\hat{\phi}_{z, j}$, $f_{z, j}$, and $g_{z, j}$.
- Random times $\sigma_{z, j}^*$, $\bar{\sigma}_{z, j}^*$, $\hat{\tau}_{z, j}^*$, and $\widetilde{\tau}_{z, j}^*$.

- Points $\tilde{x}_{z,j}$, $\tilde{y}_{z,j}$, and $\tilde{m}_{z,j}$.
- Curves $\tilde{\eta}_{z,j}$.

For $j \in \mathbf{N}$, let $\hat{\psi}_{z,j}$ be the conformal map from $\mathbf{D} \setminus (\eta_{z,j}^{T^*} \cup \bar{\eta}_{z,j}^{\bar{T}^*})$ to \mathbf{D} which fixes 0 and whose derivative at 0 has the same argument as $\Phi'_{z,j}(z)$ (the latter map is defined in Lemma 6.15).

Let $\sigma_{z,j}^*$ and $\bar{\sigma}_{z,j}^*$ be the stopping times for $\eta_{z,j}^0$ corresponding to $\tau_{z,j}^*$ and $\bar{\tau}_{z,j}^*$ (equivalently to $T_{z,j}^*$ and $\bar{T}_{z,j}^*$). Let $\phi_{z,j}^0$ be the conformal map from the connected component of $\mathbf{D} \setminus ((\eta_{z,j}^0)^{\sigma_{z,j}^*} \cup (\bar{\eta}_{z,j}^0)^{\bar{\sigma}_{z,j}^*})$ containing 0 to \mathbf{D} which takes $x_{z,j}^*$ to $-i$, $y_{z,j}^*$ to i , and the midpoint $m_{z,j}^*$ of $[x_{z,j}^*, y_{z,j}^*]_{\partial \mathbf{D}}$ to 1.

For $j = 1$, the map $f_{z,1}$ has already been defined in Section 6.2. For $j \geq 2$, we let $f_{z,j}$ be the conformal map which takes $\Phi_{z,j-1}(z)$ to 0 with $\Phi'_{z,j-1}(z) > 0$. Observe that $\hat{\psi}_{z,j-1} = f_{z,j} \circ \Phi_{z,j-1}$ (where here we take $\Phi_{z,0}$ to be the identity map and $\hat{\psi}_{z,j} = f_{z,j}$ in the case $j = 1$). For $j \geq 1$, let $\tilde{\eta}_{z,j}$ be the image under $\hat{\psi}_{z,j-1}$ of the part of η between $\eta(T_{z,j-1}^*)$ and $\bar{\eta}(\bar{T}_{z,j-1}^*)$. Note that $\tilde{\eta}_{z,j}$ involves the same part of the curve as $\eta_{z,j}^0$, which is larger than the part of the curve involved in the definition of $\eta_{z,j}$. Let $\tilde{\tau}_{z,j}^*$ and $\tilde{\bar{\tau}}_{z,j}^*$ be the times for $\tilde{\eta}_{z,j}$ and $\tilde{\bar{\eta}}_{z,j}$ corresponding to the times $T_{z,j}^*$ and $\bar{T}_{z,j}^*$ for η and $\bar{\eta}$.

Let $\tilde{x}_{z,j}$ and $\tilde{y}_{z,j}$ be the start and end points for $\tilde{\eta}_{z,j}$. Let $\tilde{\phi}_{z,j}$ be the conformal map from $\mathbf{D} \setminus (\tilde{\eta}_{z,j}^{\tilde{\tau}_{z,j}^*} \cup \tilde{\bar{\eta}}_{z,j}^{\tilde{\bar{\tau}}_{z,j}^*})$ to \mathbf{D} which takes $\tilde{x}_{z,j}^+$ to $-i$, $\tilde{y}_{z,j}^-$ to i and the midpoint $\tilde{m}_{z,j}$ of $[\tilde{x}_{z,j}, \tilde{y}_{z,j}]_{\partial \mathbf{D}}$ to 1. Let $g_{z,j} : \mathbf{D} \rightarrow \mathbf{D}$ be the conformal map taking $(\tilde{\phi}_{z,j} \circ f_{z,j})(b)$ to b for $b = -i^+, i^-, 1$. Let

$$\hat{\phi}_{z,j} := g_{z,j} \circ \tilde{\phi}_{z,j} \circ f_{z,j}.$$

Observe that (with $\Phi_{z,j}$ as in Lemma 6.15)

$$\Phi_{z,j} = \hat{\phi}_{z,j} \circ \cdots \circ \hat{\phi}_{z,1}. \quad (6.17)$$

See Figure 6.6 for an illustration of these maps in the case $j = 2$ (which has all of the features of the general case).

Lemma 6.16. *If β_1 is chosen sufficiently large, then on the event $E_n(z)$, we have*

$$|\hat{p}'_{z,n}(z)| \asymp |\hat{\psi}'_{z,n}(z)|, \quad (6.18)$$

with the implicit constants independent of n and uniform for $z \in B_d(0)$.

We refer to Figure 6.7 for an illustration of some of the maps involved in the proof of Lemma 6.16.

Proof of Lemma 6.16. Assume we are working on the event $E_n(z)$. Let $p_{z,n-1}^*$ be the conformal map from $\hat{\psi}_{z,n}(\hat{D}_{z,n-1})$ to \mathbf{D} with $p_{z,n-1}^*(0) = 0$ and $(p_{z,n-1}^*)'(0) > 0$ (in the case $n = 1$, we take $p_{z,n-1}^*$ to be the identity). Let $p_{z,n}^{**}$ be the conformal map from $(p_{z,n-1}^* \circ \hat{\psi}_{z,n})(\hat{D}_{z,n})$ to \mathbf{D} with $p_{z,n}^{**}(0) = 0$ and $\arg(p_{z,n}^{**})'(0)$ chosen in such a way that

$$\hat{p}_{z,n} = p_{z,n}^{**} \circ p_{z,n-1}^* \circ \hat{\psi}_{z,n}. \quad (6.19)$$

By the Beurling estimate and [Law05, Exercise 2.7] the diameter of $\mathbf{D} \setminus \hat{\psi}_{z,n}(\hat{D}_{z,n-1})$ tends uniformly to 0 as $\beta_n \rightarrow \infty$ (and hence also as $\beta_1 \rightarrow \infty$). Therefore, if β_1 is chosen sufficiently large, then $|(p_{z,n-1}^*)'(0)| \asymp 1$.

Let $\psi_{z,n}^F$ be as in the definition of $F_{z,n}$. The set $(p_{z,n-1}^* \circ \hat{\psi}_{z,n})(\partial \hat{D}_{z,n})$ is the image of $\psi_{z,n}^F(\partial D_{z,n})$ under a conformal map which fixes 0 and maps the complement of the set $\psi_{z,1}^F(\eta_{z,1}([\tau_{z,1}, \tau_{z,1}^*])) \cup \psi_{z,1}^F(\bar{\eta}_{z,1}([\bar{\tau}_{z,1}, \bar{\tau}_{z,1}^*]))$ to \mathbf{D} . By condition 2 in the definition of $E_{z,n}$, the distance from 0 to $(p_{z,n-1}^* \circ \hat{\psi}_{z,n})(\partial \hat{D}_{z,n})$ is proportional to the distance from 0 to $\psi_{z,n}^F(\partial D_{z,n})$. By condition 1 in the definition of $F_{z,n}$, this distance is $\asymp 1$. Consequently, $|(p_{z,n}^{**})'(0)| \asymp 1$ so our desired result follows from (6.19). \square

Lemma 6.17. *Let $\zeta \in (0, a/100)$. If we make the parameter r in the definition of $E_{z,j}$ sufficiently small (depending on ζ and the other parameters but not on (β_j) or (u_j) and uniform for $z \in B_d(0)$) and β_1 is sufficiently large (in a manner which does not depend on (u_j) and is uniform for $z \in B_d(0)$) then for any sub-arc I of $[\tilde{x}_{z,n+1}, \tilde{y}_{z,n+1}]_{\partial \mathbf{D}}$ lying at distance at least ζ from $\tilde{x}_{z,n+1}$ and $\tilde{y}_{z,n+1}$, we have that $\hat{\phi}_{z,n+1}$ is Lipschitz on I and $\hat{\phi}_{z,n+1}^{-1}$ is Lipschitz on $\tilde{\phi}_{z,n+1}(I)$ on the event $E_n(z)$ with Lipschitz constants independent of (β_j) and (u_j) and uniform for $z \in B_d(0)$.*

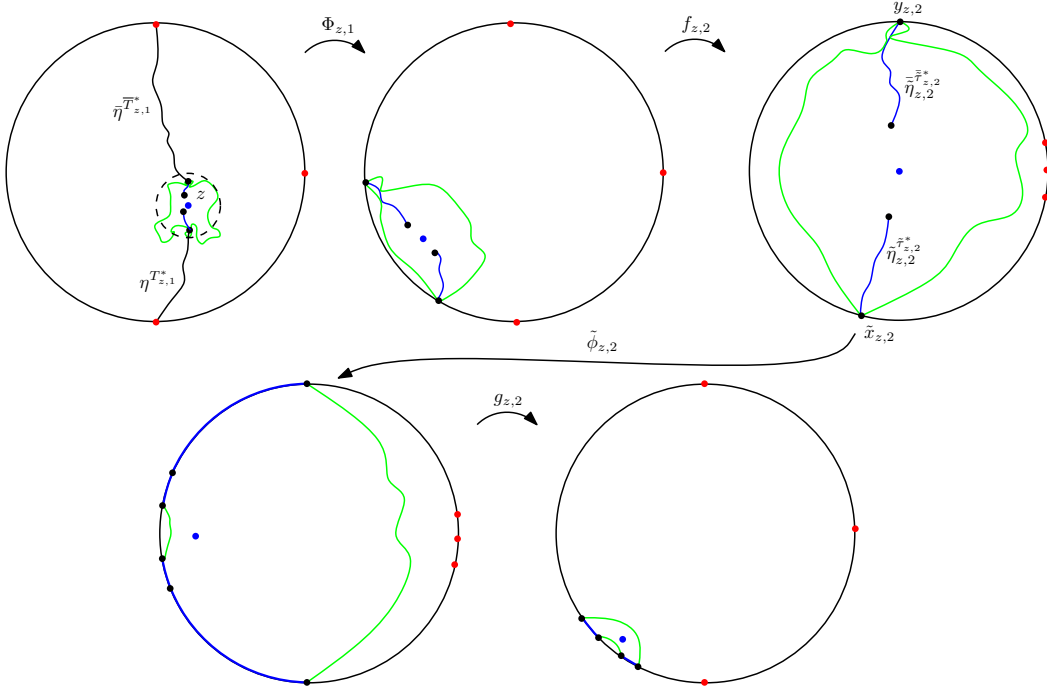


Figure 6.6: An illustration of some of the maps associated with the events $E_{z,1}$ and $E_{z,2}$. The images of $-i$, i , and 1 are shown in red. The images of z are shown in blue. The black curves are associated with the event $E_{z,1}$ and the blue curves are associated with the event $E_{z,2}$. The map $\hat{\phi}_{z,2}$ is the composition of the last four maps in the figure. The map $\Phi_{z,2}$ is the composition of all four maps.

Proof. See Figure 6.7 for an illustration of the argument. Throughout, we work on the event $E_n(z)$.

Let $A := \hat{\psi}_{z,n}((\hat{\eta}_{z,n}^+)^{t_{z,n}^+})$. Then A disconnects $\tilde{\eta}_{z,n+1}$ from I in $\partial\mathbf{D}$. We claim that there is a constant $\delta > 0$, independent of (β_j) and (u_j) and uniform for $z \in B_d(0)$, such that if β_1 (and hence β_j) is sufficiently large, then

$$E_{z,n-1} \cap E_{z,n} \subset \{\text{dist}(A, I) \geq \delta\}. \quad (6.20)$$

Given the claim, the statement of the lemma follows from Lemma 2.8.

Let $\psi_{z,n}^*$ be a conformal map from the connected component of $\mathbf{D} \setminus (\eta_{z,n}^{\tau_{z,n}^*} \cup \bar{\eta}_{z,n}^{\bar{\tau}_{z,n}^*})$ containing 1 on its boundary to \mathbf{D} which fixes 0 . This map is defined only up to a rotation, which we will specify shortly. Let $\eta_{z,n+1}^*$ be the image under $\psi_{z,n}^*$ of the part of $\eta_{z,n}$ between $\eta_{z,n}(\tau_{z,n}^*)$ and $\bar{\eta}_{z,n}(\bar{\tau}_{z,n}^*)$. We can normalize $\psi_{z,n}^*$ in such a way that we have

$$\eta_{z,n+1}^* = p_{z,n-1}^*(\tilde{\eta}_{z,n+1}),$$

with $p_{z,n-1}^*$ as in the proof of Lemma 6.16.

By condition 2 in the definition of $\tilde{E}_{z,n}$ and condition 2 in the definition of $E_{z,n}$, we have that $\mathbf{D} \setminus \hat{\psi}_{z,n}(\hat{D}_{z,n-1})$ lies at distance at least a positive constant depending only on a from $\tilde{x}_{z,n}$ and $\tilde{y}_{z,n}$ on $E_{z,n}$. Since the diameter of $\mathbf{D} \setminus \hat{\psi}_{z,n}(\hat{D}_{z,n-1})$ tends to 0 as $\beta_n \rightarrow \infty$ (by the argument of Lemma 6.16), we have that $(p_{z,n-1}^*)^{-1}$ is nearly constant near $\tilde{x}_{z,n}$ and $\tilde{y}_{z,n}$ if β_1 is large. By the Schwarz lemma $(p_{z,n-1}^*)^{-1}$ decreases distances to $\partial\mathbf{D}$. Hence the distance from A to I is at least a β_1 -independent constant times the distance from A^* to I if β_1 is large, where

$$A^* = p_{z,n-1}^*(A) = \psi_{z,n}^*((\eta_{z,n}^+)^{t_{z,n}^+}).$$

Hence it is enough to prove (6.20) with A^* in place of A .

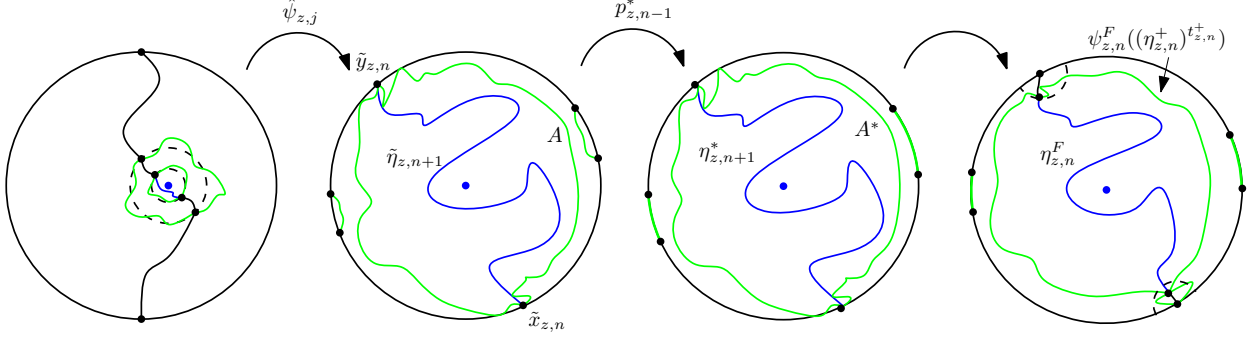


Figure 6.7: An illustration of the maps used in the proof of Lemma 6.17. In order to control the distance from $\tilde{\eta}_{z,n+1}$ to an arc on the right boundary of the disk, we compare $\tilde{\eta}_{z,n+1}$ to the curve $\eta_{z,n+1}^*$ and then to the curve $\eta_{z,n}^F$. The distance from the last curve to an appropriate arc of the right boundary is bounded below by condition 5 in the definition of $F_{z,n}$.

Let $I' \supset I$ be a slightly larger arc. By condition 2 in the definition of $E_{z,n}$ the distance from A^* to I is \asymp the distance from $\psi_{z,n}^F((\eta_{z,n}^+)^{t_{z,n}^+})$ to I' if r is chosen sufficiently small, where $\psi_{z,n}^F$ is as in the definition of $F_{z,n}$. We conclude by applying condition 5 in the definition of $F_{z,n}$. \square

Lemma 6.18. *We can choose the parameters in Section 6.2 independently of n and the parameter β_1 in such a way that on $E_n(z)$ we have*

$$e^{-\beta_n(q+u_n)} \preceq |\phi'(w)| \preceq e^{-\beta_n(q-u_n)} \quad (6.21)$$

where the pair (ϕ, w) is any one of $(\phi_{z,n}, 0)$, $(\phi_{z,n}^0, 0)$, $(\tilde{\phi}_{z,n}, 0)$, or $(\hat{\phi}_{z,n}, \Phi_{z,n-1}(z))$. The implicit constants are independent of n and uniform for z in compacts.

Proof. By definition of $E_{z,n}$ (in particular, of $\tilde{E}_{z,n}$), the statement of the lemma is true for $(\phi, w) = (\phi_{z,n}, 0)$. We will now transfer the estimate (6.21) from $\phi_{z,n}$ to $\phi_{z,n}^0$ to $\tilde{\phi}_{z,n}$ to $\hat{\phi}_{z,n}$. This latter map is our primary interest, mostly because of (6.17). Throughout, we assume that $E_{z,n-1} \cap E_{z,n}$ occurs and require all implicit constants to be independent of m and n and uniform for z in compacts.

Let $\phi_{z,n}^{**}$ be the conformal map from the connected component of $\mathbf{D} \setminus (\eta_{z,n}^* \cup \bar{\eta}_{z,n}^*)$ containing 0 to \mathbf{D} which takes $-i^+$ to $-i$, i^- to i , and 1 to 1. Let $g_{z,n}^*$ be the conformal automorphism of \mathbf{D} which fixes $-i$ and i and takes $(\phi_{z,n}^{**} \circ \psi_{z,n})(m_{z,n}^*)$ to 1. Then we have

$$\phi_{z,n}^0 = g_{z,n}^* \circ \phi_{z,n}^{**} \circ \psi_{z,n} \quad (6.22)$$

By condition 2 in the definition of $E_{z,n}$ and Lemma B.4, we have $|(\phi_{z,n}^{**})'(0)| \asymp |\phi_{z,n}'(0)|$ provided r is chosen sufficiently small independently of n and uniformly for z in compacts. Furthermore, by Lemma 2.8 and condition 4 in the definition of $E_{\beta_n}^{q;u_n}(\cdot)$, we have that $\mathcal{G}(\phi_{z,n}^{**}, \mu')$ holds on $E_{z,n}$ for some $\mu' \in \mathcal{M}$ depending only on μ . By condition 1 in the definition of $L_{z,n}$ and condition 4 in the definition of $F_{z,n-1}$, $|x_{z,n}^* - y_{z,n}^*| \geq 1$ on $E_{z,n-1} \cap E_{z,n}$. By combining this with condition 6 in the definition of $L_{z,n}$ we have $|(g_{z,n}^*)'(z)| \asymp 1$ on all of \mathbf{D} on the event $E_{z,n}$. Hence (6.22) implies (6.21) for $\phi_{z,n}^0$.

By Lemma B.3 applied with $U = \mathbf{D} \setminus ((\tilde{\eta}_{z,n})^* \cup (\tilde{\eta}_{z,n})^*)$, $D = \hat{\psi}_{z,n-1}(\hat{D}_{z,n-1})$, $\phi = \tilde{\phi}_{z,n}$, $\hat{\phi} = \phi_{z,n}^0$, and $z = \hat{z} = 0$, the estimate (6.21) for $\phi_{z,n}^0$ implies the estimate (6.21) for $\tilde{\phi}_{z,n}$. Note that the conditions of Lemma B.3 (with parameters $\zeta, \Delta, \delta \geq 1$) follow easily from the definition of $E_n(z)$ together with Lemmas 6.16 and 6.17.

To get the estimate for $\hat{\phi}_{z,n}$, write

$$|\hat{\phi}_{z,n}'(\Phi_{z,n-1}(z))| = |g_{z,n}'(\tilde{\phi}_{z,n}(0))| |\tilde{\phi}_{z,n}'(0)| |f_{z,n}'(\Phi_{z,n-1}(z))|. \quad (6.23)$$

By the Koebe quarter theorem, $|f'_{z,n}(\Phi_{z,n-1}(z))| \asymp (1 - |\Phi_{z,n-1}(z)|)^{-1}$. In fact, we have $|f'_{z,n}(w)| \asymp 1 - |\Phi_{z,n-1}(z)|$ on subsets of \mathbf{D} at positive distance from $\Phi_{z,n-1}(z)$.

By condition 2 in the definition of $E_{\beta_{n-1}}^{q;u_{n-1}}(\cdot)$, we can find $\zeta > 0$ depending only on a such that $f_{z,n}([-i, i]_{\partial\mathbf{D}})$ lies at distance at least ζ from $\tilde{x}_{z,n}$ and $\tilde{y}_{z,n}$ on $E_{z,n-1}$. By Lemma 6.17, on $E_{z,n-1}$, it holds that $\tilde{\phi}_{z,n}$ distorts the distances between points in $f_{z,n}([-i, i]_{\partial\mathbf{D}})$ by at most a constant factor. Since conformal automorphisms of \mathbf{D} depend smoothly on their parameters, it follows that

$$|g'_{z,n}| \asymp |(f_{z,n}^{-1})'| \quad (6.24)$$

on the whole left half of \mathbf{D} . By combining this with (6.23) and the first statement of the lemma we conclude. \square

Proof of Lemma 6.15. Assume $E_n(z)$ holds and that β_1 has been chosen so that the conclusion of Lemma 6.18 holds. Assertion 1 is immediate from Lemma 6.18 and the relation (6.17). Note that we can absorb the implicit constants in (6.21) into an additional factor of $e^{\bar{u}_n}$ due to condition 6 of Lemma 6.14.

To prove assertion 2, we induct on n . The case $n = 1$ is immediate from the definitions of the events. Now suppose $n \geq 2$ and the result has been proven with n replaced by $n - 1$. Since $\hat{\psi}_{z,n-1}^{-1}$ maps $\mathbf{D} \setminus (\tilde{\eta}_{z,n}^{*,n} \cup \tilde{\eta}_{z,n}^{\bar{*},n})$ to $\mathbf{D} \setminus (\eta_{z,n}^{T*,n} \cup \bar{\eta}_{z,n}^{\bar{T}*,n})$ and fixes 0, the Koebe quarter theorem implies

$$\text{dist}\left(z, \eta_{z,n}^{T*,n} \cup \bar{\eta}_{z,n}^{\bar{T}*,n}\right) \asymp |(\hat{\psi}_{z,n-1}^{-1})'(0)| \text{dist}\left(0, \tilde{\eta}_{z,n}^{*,n} \cup \tilde{\eta}_{z,n}^{\bar{*},n}\right). \quad (6.25)$$

By a second application of the Koebe quarter theorem,

$$|(\hat{\psi}_{z,n-1}^{-1})'(0)| \asymp \text{dist}\left(z, \eta_{z,n-1}^{T*,n-1} \cup \bar{\eta}_{z,n-1}^{\bar{T}*,n-1}\right). \quad (6.26)$$

By the inductive hypothesis,

$$e^{-\bar{\beta}_{n-1}-\lambda(n-1)} \preceq \text{dist}\left(z, \eta_{z,n-1}^{T*,n-1} \cup \bar{\eta}_{z,n-1}^{\bar{T}*,n-1}\right) \preceq e^{-\bar{\beta}_{n-1}+\lambda(n-1)}. \quad (6.27)$$

By Lemma 6.16 (applied with n replaced by $n - 1$), we have

$$\text{dist}\left(0, \tilde{\eta}_{z,n}^{*,n} \cup \tilde{\eta}_{z,n}^{\bar{*},n}\right) \asymp \text{dist}\left(0, (\eta_{z,n}^0)^{\sigma_{z,n}^*} \cup (\bar{\eta}_{z,n}^0)^{\bar{\sigma}_{z,n}^*}\right). \quad (6.28)$$

By definition of $E_{z,n}$, this last distance is $\asymp e^{-\beta_n}$ on $E_{z,n}$. Provided λ is chosen sufficiently large, independently of n and $z \in B_d(0)$, we can now complete the induction by combining (6.25), (6.26), (6.27), and (6.28).

By condition 2 in the definition of the event from Section 6.1 and condition 2 in the definition of $E_{z,n}$, if we choose r sufficiently small relative to a then the harmonic measure from z of each of the two sides of $\eta_{z,n}^{T*,n}$ (resp. each of the two sides of $\bar{\eta}_{z,n}^{\bar{T}*,n}$) in $\mathbf{D} \setminus (\eta_{z,n}^{T*,n} \cup \bar{\eta}_{z,n}^{\bar{T}*,n})$ is at least some constant which does not depend on n or the particular choice of $z \in B_d(0)$. By the Beurling estimate this implies assertion 3.

For assertion 4, we use assertion 2 and the Koebe quarter theorem to see that there exists radii $\rho' > \rho > 0$ such that $\rho \preceq e^{-\bar{\beta}_{n-1}-\lambda n}$, $\rho' \preceq e^{-\bar{\beta}_{n-1}+\lambda n}$, $\hat{\psi}_{z,n-1}^{-1}(\mathcal{B}_{\beta_n+\log 2}) \supset B_{\rho}(z)$, and $\hat{\psi}_{z,n-1}^{-1}(\mathcal{B}_{\beta_n-\log 2}) \subset B_{\rho'}(z)$. By combining this with condition 1 in the definition of $F_{z,j}$ we see that assertion 4 holds (after possibly increasing λ). \square

6.5 Probabilistic properties

In this subsection we study the events defined above from a probabilistic perspective, and eventually prove our two-point estimate.

Proposition 6.19. *Let $z, w \in B_d(0)$. Let λ be as in assertion 2 of Lemma 6.15. Choose $k \in \mathbf{N}$ such that $e^{-\bar{\beta}_{k+1}-\lambda(k+1)} \leq |z - w| \leq e^{-\bar{\beta}_k-\lambda k}$. Then for any $n \in \mathbf{N}$ with $\bar{\beta}_n - \lambda n \geq \bar{\beta}_{k+1} + \lambda(k+2)$,*

$$\mathbf{P}(E_n(z) \cap E_n(w)) \preceq e^{\bar{\beta}_k o_k(1)} \frac{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))}{\mathbf{P}(E_k(w))} \quad (6.29)$$

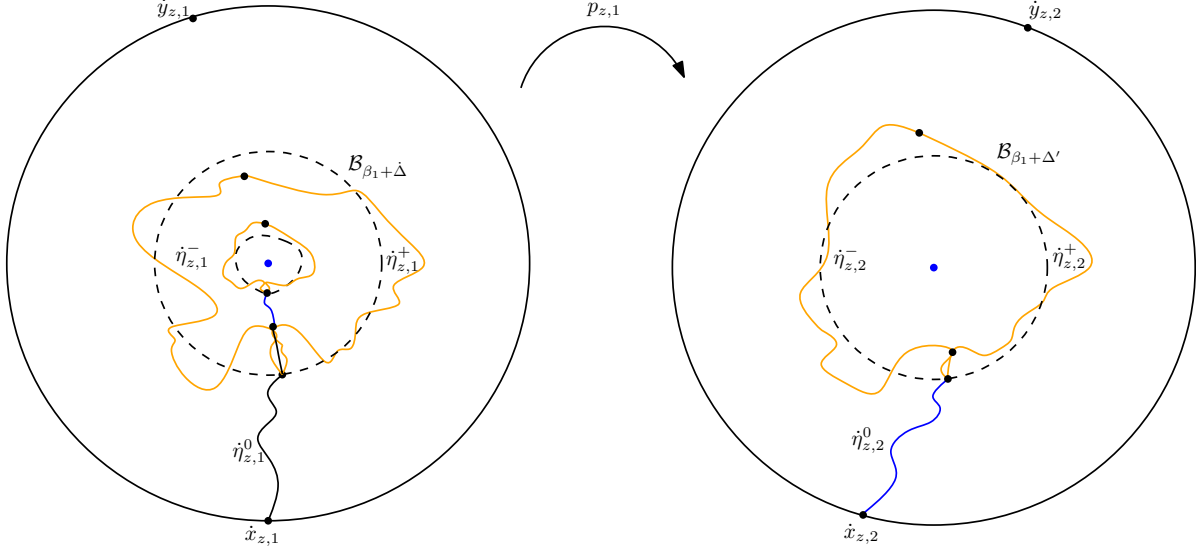


Figure 6.8: An illustration of two stages of the construction of the pockets $\dot{D}_{z,j}$. The pocket $\dot{D}_{z,1}$ is the region containing the origin (shown in blue) surrounded by the outermost orange curves in the first figure. The pocket $\dot{D}_{z,2}$ is the region containing the origin surrounded by the orange curves in the second figure. Since the orange flow lines are started at stopping times for $\eta_{z,j}^0$, we can easily compute the conditional law of the objects involved in this construction using the result of [MS16a].

with the implicit constants independent of n and k , the $o_k(1)$ independent of n , and both deterministic and uniform for $z, w \in B_d(0)$.

Remark 6.20. In the setting of Proposition 6.19 we have

$$e^{-\bar{\beta}_k} = |z - w|^{1+o_{|z-w|}(1)}$$

so by Lemma 6.25 below we can rewrite the estimate (6.29) as

$$\mathbf{P}(E_n(z) \cap E_n(w)) \leq |z - w|^{-\gamma^*(q) + o_{|z-w|}(1)} \mathbf{P}(E_n(z)) \mathbf{P}(E_n(w)).$$

This is the form of the estimate we will use when we prove lower bounds for the Hausdorff dimensions of our sets.

In order to prove Proposition 6.19, we first need to introduce an additional family of auxiliary flow lines which form the same pocket as $\eta_{z,n}^\pm$ on the event $E_n(z)$, but whose law is easier to analyze.

To this end, let $\dot{\eta}_{z,1}^0 = \eta_{z,1}^0$. Let $\dot{\tau}_{z,1}$ be the first time $\dot{\eta}_{z,1}^0$ hits $\mathcal{B}_{\beta_1+\Delta}$ (here Δ is as in the definition of $L_{z,1}$). Let $\dot{h}_{z,1} = h \circ f_{z,1}^{-1} - \chi \arg(f_{z,1}^{-1})'$. Let $\dot{\eta}_{z,1}^\pm$ be the flow lines of $\dot{h}_{z,1}$ started from $\dot{\tau}_{z,1}$ with angles $\mp\theta$. Let $\dot{D}_{z,1}$ be the connected component of $\mathbf{D} \setminus (\dot{\eta}_{z,1}^+ \cup \dot{\eta}_{z,1}^-)$ which contains the origin. Let $\dot{p}_{z,1}$ be the conformal map from $\dot{D}_{z,1}$ to \mathbf{D} with $\dot{p}_{z,1}(0) = 0$ and $\dot{p}'_{z,1}(0) > 0$.

Inductively, suppose $j \geq 2$ and that $\dot{\eta}_{z,j-1}$, $\dot{\tau}_{z,j-1}$, $\dot{h}_{z,j-1}$, $\dot{\eta}_{z,j-1}^\pm$, $\dot{D}_{z,j-1}$, and $\dot{p}_{z,j-1}$ have been defined. Let $\dot{\eta}_{z,j}^0 := \dot{p}_{z,j-1}(\dot{\eta}_{z,j-1}^0 \cap \dot{D}_{z,j-1})$. Let $\dot{\tau}_{z,j}$ be the first time $\dot{\eta}_{z,j}^0$ hits $\mathcal{B}_{\beta_1+\Delta}$. Let $\dot{h}_{z,j} = \dot{h}_{z,j-1} \circ p_{z,j-1}^{-1} - \chi \arg(p_{z,j-1}^{-1})'$. Let $\dot{\eta}_{z,j}^\pm$ be the flow lines of $\dot{h}_{z,j}$ started from $\dot{\tau}_{z,j}$ with angles $\mp\theta$. Let $\dot{D}_{z,j}$ be the connected component of $\mathbf{D} \setminus (\dot{\eta}_{z,j}^+ \cup \dot{\eta}_{z,j}^-)$ which contains the origin. Let $\dot{p}_{z,j}$ be the conformal map from $\dot{D}_{z,j}$ to \mathbf{D} with $\dot{p}_{z,j}(0) = 0$ and $\dot{p}'_{z,j}(0) > 0$. See Figure 6.5 for an illustration of this construction.

Let

$$\hat{\eta}_{z,j}^\pm := (f_{z,1}^{-1} \circ \dot{p}_{z,1}^{-1} \circ \cdots \circ \dot{p}_{z,j-1}^{-1})(\dot{\eta}_{z,j}^\pm) \quad \text{and} \quad \hat{D}_{z,j} := (f_{z,1}^{-1} \circ \dot{p}_{z,1}^{-1} \circ \cdots \circ \dot{p}_{z,j-1}^{-1})(\dot{D}_{z,j})$$

be, respectively, the flow lines for h corresponding to $\dot{\eta}_{z,j}^\pm$ and the pocket they form surrounding z .

Our interest in the above objects stems from the following lemma.

Lemma 6.21. *Define the objects $\dot{\eta}_{z,j}^0$, $\dot{D}_{z,j}$, etc. as above, and retain the notation of Section 6.2. If β_1 (and hence each β_j for $j \geq 1$) is chosen sufficiently large, depending only on the parameter \dot{r} , then the following is true.*

1. The field $\dot{h}_{z,j}$ is conditionally independent from $h|_{\mathbf{D} \setminus \hat{D}_{z,j-1}}$ given $\hat{D}_{z,j}$ and $h|_{\partial \hat{D}_{z,j-1}}$.
2. Let $\dot{x}_{z,j}$ and $\dot{y}_{z,j}$ be the start and end points of $\dot{\eta}_{z,j}^0$. Conditional on $\hat{D}_{z,j-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,j-1}}$, we have that $\dot{\eta}_{z,j}^0$ is the zero-angle flow line of $\dot{h}_{z,j}$ from $\dot{x}_{z,j}$ to $\dot{y}_{z,j}$. In particular, the law of $\dot{\eta}_{z,j}^0$ under this conditioning is that of a chordal $\text{SLE}_\kappa(\rho^1; \rho^1)$ from $\dot{x}_{z,j}$ to $\dot{y}_{z,j}$ in \mathbf{D} , with ρ^1 as in (6.12).
3. On the event $E_n(z)$, we a.s. have $\eta_{z,j}^0 = \dot{\eta}_{z,j}^0$ for each $j \leq n+1$ and $\hat{D}_{z,j} = \dot{D}_{z,j}$ for each $j \leq n$.

Proof. To obtain assertion 1, we start by observing that by [MS16a, Theorem 1.1] and induction, for each $j \geq 2$, the set $\dot{A}_{z,j-1} := \eta^{\dot{T}_{z,j-1}} \cup \hat{\eta}_{z,j-1}^+ \cup \hat{\eta}_{z,j-1}^-$ is a local set for h in the sense of [SS13, Section 3.3]. Hence the conditional law of h given $\dot{A}_{z,j-1}$ and $h|_{\dot{A}_{z,j-1}}$ in each complementary connected component of $\mathbf{D} \setminus \dot{A}_{z,j-1}$ is independently that of a zero-boundary GFF plus a certain $(\dot{A}_{z,j-1}, h|_{\dot{A}_{z,j-1}})$ -measurable harmonic function. In particular, the conditional law of $\dot{h}_{z,j-1}$ given $\dot{A}_{z,j-1}$ is that of a GFF on \mathbf{D} with boundary data $\lambda - \theta\chi - \chi \cdot \text{winding}$ on $[\dot{y}_{z,j}, \dot{x}_{z,j}]_{\partial \mathbf{D}}$ and $-\lambda + \theta\chi$ on $\lambda - \theta\chi - \chi \cdot \text{winding}$ on $[\dot{x}_{z,j}, \dot{y}_{z,j}]_{\partial \mathbf{D}}$, where λ and χ are as in Section 2.5 and the term “winding” has the meaning of [MS16a, Figure 1.9].

By [MS16a, Theorem 1.2] and locality, $\dot{A}_{z,j-1}$ is a.s. determined by $\hat{D}_{z,j-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,j-1}}$. Hence we get the same conditional law for $\dot{h}_{z,j}$ if we instead condition on $\hat{D}_{z,j-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,j-1}}$. Since this law depends only on $\hat{D}_{z,j-1}$ and $h|_{\partial \hat{D}_{z,j-1}}$, we obtain assertion 1.

Assertion 2 follows immediately from our description of the conditional law of $\dot{h}_{z,j}$ and [MS16a, Theorems 1.1 and 2.4].

It remains to prove assertion 3. See Figure 6.5 for an illustration. By the Koebe distortion theorem, we have that as $\beta_j \rightarrow \infty$, $e^{\beta_j + \Delta} \text{diam } \psi_{z,j}(\mathcal{B}_{\beta_j + \Delta})$ and $e^{\beta_j + \Delta} \text{dist}(0, \partial \psi_{z,j}(\mathcal{B}_{\beta_j + \Delta}))$ tend uniformly to $|\psi'_{z,j}(0)|^{-1}$. Thus if β_1 (and hence β_j) is chosen sufficiently large, then by condition 5 in the definition of $L_{z,j}$ we have

$$\mathcal{B}_{\beta_j + \dot{r}} \subset \psi_{z,j}(\mathcal{B}_{\beta_j + \Delta}) \subset \mathcal{B}_{\beta_j}. \quad (6.30)$$

By (6.30) and condition 2 in the definition of $F_{z,1}$, it follows that on $E_1(z)$, the starting point for the curves $\psi_{z,1}(\dot{\eta}_{z,1}^\pm)$ is disconnected from $\partial \mathbf{D}$ by the parts of $\eta_{z,1}^\pm$ traced before they hit $b_{z,1}$. Therefore, $\eta_{z,1}^\pm$ a.s. hit and (by [MS16a, Theorem 1.5]) subsequently merge with $\psi_{z,1}(\dot{\eta}_{z,1}^\pm)$ before reaching $b_{z,1}$. This proves assertion 3 in the case $n = 1$. The general case follows from (6.30) and induction. \square

Note that Lemma 6.21 implies in particular that the conditional law of $\eta_{z,n}^0$ given $\hat{D}_{z,n-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,n-1}}$ on the event $E_{n-1}(z)$ is that of a chordal $\text{SLE}_\kappa(\rho^1; \rho^1)$ process from $x_{z,n}$ to $y_{z,n}$ in \mathbf{D} . Furthermore, $\eta_{z,n}^0$ is equal to a flow line of $h|_{\hat{D}_{z,n-1}}$ on this event. Since the parts of η and the auxiliary flow lines which generated $\mathcal{F}_{z,n-1}$ all lie outside $\hat{D}_{z,n-1}$, [MS16a, Theorem 1.2] implies that $\mathcal{F}_{z,n-1}$ is a.s. determined by $\hat{D}_{z,n-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,n-1}}$ on the event $E_{n-1}(z)$.

For $n \geq 1$, define the event $\dot{L}_{z,n}$ and the curve $\dot{\eta}_{z,n}$ in the same manner as the event $L_{z,n}$ and the curve $\eta_{z,n}$ but with $\dot{\eta}_{z,n}^0$ in place of $\eta_{z,n}^0$. Also fix $\dot{a} \in (0, 2)$ and let

$$\dot{K}_{z,n} = \dot{K}_{z,n}(\dot{a}) := \{|\dot{x}_{z,n+1} - \dot{y}_{z,n+1}| \geq \dot{a}\}. \quad (6.31)$$

We can now prove the analogue of Lemma 6.8 for $n \geq 2$.

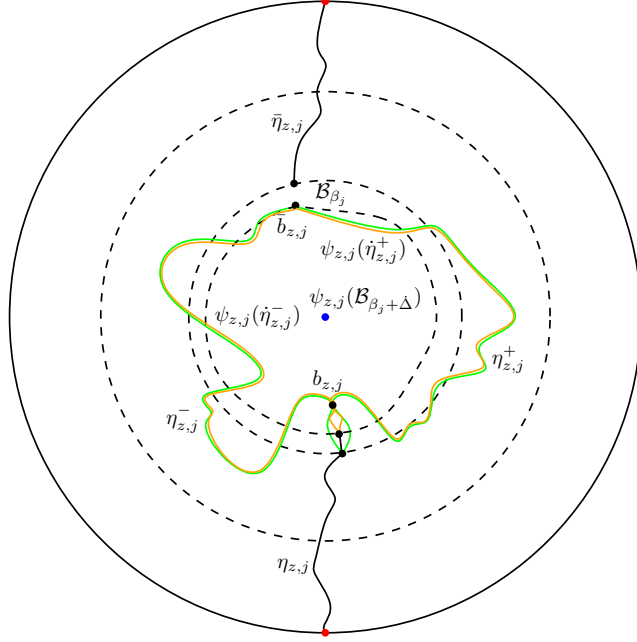


Figure 6.9: An illustration of the proof of assertion 3 of Lemma 6.21. The inner dotted quasi-circle is the image of $\mathcal{B}_{\beta_j+\Delta}$ under $\psi_{z,j}$. On the event $E_j(z)$, the orange flow lines $\psi_{z,j}(\eta_{z,j}^\pm)$ quickly merge with $\eta_{z,j}^\pm$ and form the same pocket around zero as the flow lines $\eta_{z,j}^\pm$.

Lemma 6.22. *Suppose we are in the setting of Lemma 6.21. If the parameters involved in the definition of $L_{z,1}$ are chosen appropriately, independently of n , β_n , u_n , and $z \in B_d(0)$ then we can find a deterministic $p > 0$ which does not depend on n , β_n , u_n , and $z \in B_d(0)$ such that for each $n \geq 2$, we have*

$$\mathbf{P}(L_{z,n} | \mathcal{F}_{z,n-1}) \mathbf{1}_{E_{n-1}(z)} \geq p \mathbf{1}_{E_{n-1}(z)}, \quad \mathbf{P}(\dot{L}_{z,n} | \hat{D}_{z,n-1}, h|_{\mathbf{D} \setminus \hat{D}_{z,n-1}}) \mathbf{1}_{\dot{K}_{z,n-1}} \geq p \mathbf{1}_{\dot{K}_{z,n-1}}.$$

Proof. We prove only the statement for $L_{z,n}$. The statement for $\dot{L}_{z,n}$ is proven in exactly the same manner. Let A be the event that the part of $\eta_{z,n}^0$ lying between $\eta_{z,n}^0(\sigma_{z,1})$ and $\bar{\eta}_{z,1}^0(\bar{\sigma}_{z,1})$ never exits $\mathcal{B}_{\tilde{\Delta}}$. Let B be the event that all of the conditions in the definition of $L_{z,n}$ except possibly condition 7 occur.

In light of Lemma 6.21, we can apply [MW14, Lemma 2.5] to get that if the parameters are chosen appropriately, independently of n , then $\mathbf{P}(A \cap B | \mathcal{F}_{z,n-1}) \mathbf{1}_{E_{n-1}(z)} \geq \mathbf{1}_{E_{n-1}(z)}$ (here we use condition 4 in the definition of $F_{z,n-1}$ to get that $x_{z,n}$ and $y_{z,n}$ lie at uniformly positive distance from one another on $E_{n-1}(z)$). Hence $\mathbf{P}(A | B, \mathcal{F}_{z,n-1}) \mathbf{1}_{E_{n-1}(z)} \geq 1$. It follows that there exists a sub-event of B with uniformly positive probability and a $p_L > 0$ on which $\mathbf{P}(A | (\eta_{z,n}^0)^{\sigma_{z,n}} \cup (\bar{\eta}_{z,n}^0)^{\bar{\sigma}_{z,n}}, \mathcal{F}_{z,n-1}) \mathbf{1}_{E_{n-1}(z)} \geq p_L \mathbf{1}_{E_{n-1}(z)}$. On this sub-event, $L_{z,n}$ occurs. \square

For $n \in \mathbf{N}$ and $j \in \mathbf{N}$, define flow lines $\dot{\eta}_{z,j}^{n,\pm}$, times $\dot{t}_{z,j}^{n,\pm}$ and $\dot{t}_{z,j}^{n,\pm}$, and events $\dot{E}_{z,j}^n$ in the same manner as $\eta_{0,j}^\pm$, $\tilde{t}_{0,j}^\pm$, $t_{0,j}^\pm$, and $E_{0,j}$ but with h replaced by $\dot{h}_{z,n}$, $\eta_{0,1}^0 = \eta$ replaced by $\dot{\eta}_{z,n}^0$, and the sequence $(\beta_j, u_j)_{j \in \mathbf{N}}$ replaced by $(\beta_{n+j}, u_{n+j})_{j \in \mathbf{N}}$ (we retain the above definition of $\dot{L}_{z,n}$). Also let

$$\dot{E}_k^n(z) := \bigcap_{j=1}^k \dot{E}_{z,j}^n. \quad (6.32)$$

Lemma 6.23. *Suppose we are in the setting of Section 6.2. If the auxiliary parameters are chosen appropriately, independently of n , $(\beta_j, u_j)_{j \geq 1}$, and $z \in B_d(0)$, and if β_1 is chosen sufficiently large, then the following*

three laws are a.s. s.m.a.c. (Definition C.1) with deterministic (i.e., independent of whatever realization we are conditioning on) constants uniform in n , $(\beta_j, u_j)_{j \geq 1}$, and $z \in B_d(0)$.⁷

1. The conditional joint law of $\eta_{z,n}$ and $\left\{(\eta_{z,j}^+)^{\tilde{t}_{z,j}^+}, (\eta_{z,j}^-)^{\tilde{t}_{z,j}^-}\right\}_{j \geq n}$ given any realization of $\mathcal{F}_{z,n-1} \vee \mathcal{F}_{z,n}^0$ for which $E_{n-1}(z) \cap L_{z,n}$ occurs, restricted to the event $\mathcal{G}'(\eta_{z,n}, \mu)$.
2. The conditional joint law of $\dot{\eta}_{z,n}$ and $\left\{(\dot{\eta}_{z,j}^{n,+})^{\tilde{t}_{z,j}^{n,+}}, (\dot{\eta}_{z,j}^{n,-})^{\tilde{t}_{z,j}^{n,-}}\right\}_{j \geq 1}$ given any realization of $\hat{D}_{z,n-1}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,n-1}}$ for which $\dot{K}_{z,n-1} \cap \dot{L}_{z,n}$ occurs, restricted to the event $\mathcal{G}'(\dot{\eta}_{z,n}, \mu)$ (here $\dot{K}_{z,n-1}$ is as in (6.31)).
3. The conditional joint law of $\eta_{z,1}$ and $\left\{(\eta_{z,1}^+)^{\tilde{t}_{z,1}^+}, (\eta_{z,1}^-)^{\tilde{t}_{z,1}^-}\right\}_{j \geq 1}$ given any realization of $\mathcal{F}_{z,1}^0$ for which $L_{z,1}$ occurs, restricted to the event $\mathcal{G}'(\eta_{z,1}, \mu)$, with the sequence $(\beta_j, u_j)_{j \in \mathbf{N}}$ replaced by $(\beta_{n+j}, u_{n+j})_{j \in \mathbf{N}}$.

Proof. Throughout, we assume that all implicit constants for s.m.a.c. are deterministic, independent of n and of $(\beta_j, u_j)_{j \geq 1}$, and uniform for $z \in B_d(0)$. We will prove only the strict mutual absolute continuity of the laws 1 and 3. Strict mutual absolute continuity of the laws 1 and 2 is immediate from assertions 1 and 3 of Lemma 6.21.

Let ω be a realization of $\mathcal{F}_{z,n-1} \vee \mathcal{F}_{z,n}^0$ for which $E_{n-1}(z) \cap L_{z,n}$ occurs. Let A be the event that the part of $\eta_{z,n}^0$ between $\eta_{z,n}^0(\sigma_{z,n})$ and $\bar{\eta}_{z,n}^0(\bar{\sigma}_{z,n})$ never exits $\mathcal{B}_{\tilde{\Delta}}$. By Lemma 6.21 the conditional law of $\eta_{z,n}^0$ given any realization of $\mathcal{F}_{z,n-1}$ for which $E_{n-1}(z)$ occurs is that of an $\text{SLE}_\kappa(\rho^1; \rho^1)$ process from $x_{z,n}$ to $y_{z,n}$ in \mathbf{D} with force points located on either side of $x_{z,n}$. Since the start and end points of $\eta_{z,n}$ are equal to $-i$ and i , Lemma C.4 implies that (provided $\tilde{\Delta}$ is chosen sufficiently large) the conditional law of $\eta_{z,n}$ given ω and A is s.m.a.c. with respect to the law of a chordal SLE_κ from $-i$ to i in \mathbf{D} given (equivalently, restricted to) the event that it never leaves $\psi_{z,n}(\mathcal{B}_{\tilde{\Delta}})$. By condition 7 in the definition of $L_{z,n}$, the same is true of the conditional law of $\eta_{z,n}$ given ω , restricted to the event A . By condition 4 in the definition of $L_{z,n}$ we have $\mathcal{G}'(\eta_{z,n}, \mu) \cap L_{z,n} \subset A$, so the same is also true of the conditional law of $\eta_{z,n}$ given ω , restricted to the event $\mathcal{G}'(\eta_{z,n}, \mu)$.

By [MS16a, Theorems 1.1 and 2.4], the conditional law of $\eta_{z,n}^+$ given ω and η is that of a chordal $\text{SLE}_\kappa(\rho^0; \rho^1, \rho^2)$ from $\eta_{z,n}(\tau_{z,n})$ to $\psi_{z,n}(y_{z,n}^-)$ in the connected component of $\mathbf{D} \setminus \eta_{z,n}$ containing the right side of $-i$ on its boundary, with ρ^0, ρ^1 as in (6.12) and ρ^2 depending only on θ and κ . The force points corresponding to the weights ρ^0 and ρ^1 are located on either side of $\eta_{z,n}(\tau_{z,n})$. The extra force point corresponding to ρ^2 is located at $\psi_{z,n}(x_{z,n}^+)$.

By [MW14, Lemma 2.8] the conditional law of $(\eta_{z,n}^+)^{\tilde{t}_{z,n}^+}$ given ω and any realization of $\eta_{z,n}$ for which $\mathcal{G}'(\eta_{z,n}, \mu)$ occurs is a.s. s.m.a.c. with respect to the law of a chordal $\text{SLE}_\kappa(\rho^0; \rho^1)$ from $\eta_{z,n}(\tau_{z,n})$ to any given point v on the right boundary of the aforementioned connected component of $\mathbf{D} \setminus \eta_{z,n}$ stopped at an appropriate time. A similar statement holds for $(\eta_{z,n}^-)^{\tilde{t}_{z,n}^-}$.

Since $\eta_{z,n}^\pm$ are conditionally independent given $\eta_{z,n}^0$ and $\mathcal{F}_{z,n-1}$, it follows that the joint law of $((\eta_{z,n}^+)^{\tilde{t}_{z,n}^+}, (\eta_{z,n}^-)^{\tilde{t}_{z,n}^-})$ given ω and ω_η is a.s. s.m.a.c. with respect to the law of a pair of curves with the same description as the joint law of $((\eta_{z,1}^+)^{\tilde{t}_{z,1}^+}, (\eta_{z,1}^-)^{\tilde{t}_{z,1}^-})$ given $\eta_{z,1}$ and $\mathcal{F}_{z,1}^0$, restricted to the event $L_{z,1} \cap \mathcal{G}'(\eta_{z,1}, \mu)$, but with β_1 replaced by β_n .

It remains to deal with the conditional laws of the remaining flow lines. Let $h_{z,n+1} = h \circ \widehat{p}_{z,n}^{-1} - \chi \arg \widehat{p}_{z,n}^{-1}$. By iterative application of [MS16a, Proposition 6.1], the conditional law of $h_{z,n+1}$ given $\{\eta, (\eta_{z,n}^+)^{\tilde{t}_{z,n}^+}, (\eta_{z,n}^-)^{\tilde{t}_{z,n}^-}\}$ and $\mathcal{F}_{z,n-1}$ is that of an independent GFF in each of the complementary connected components of $\mathbf{D} \setminus \eta_{z,n+1}^0$ with boundary data depending only on θ and κ . By [MS16a, Theorem 1.2], under this conditioning the collection of curves

$$\left\{(\widehat{p}_{z,n}((\widehat{\eta}_{z,j}^+)^{\tilde{t}_{z,j}^+}), \widehat{p}_{z,n}((\widehat{\eta}_{z,j}^-)^{\tilde{t}_{z,j}^-})) : j \geq n+1\right\}$$

⁷The flow lines $\eta_{z,j}^\pm$ are only defined on the event $\eta_{z,j}$ hits a certain ball centered at the origin. We take these flow lines to be equal to a “graveyard point” in our probability space on the event that $\eta_{z,j}$ does not hit such a ball.

is a.s. determined by $h_{z,n+1}$, $\eta_{z,n}$, $(\eta_{z,n}^+)^{t_{z,n}^+}$, and $(\eta_{z,n}^-)^{t_{z,n}^-}$, in the same manner that the collection of curves

$$\left\{ (\widehat{p}_{z,1}((\widehat{\eta}_{z,j}^+)^{\widehat{t}_{z,j}^+}), \widehat{p}_{z,1}((\widehat{\eta}_{z,j}^-)^{\widehat{t}_{z,j}^-})) : j \geq 2 \right\}$$

is determined by $h_{z,2}$, $\eta_{z,1}$, $(\eta_{z,1}^+)^{t_{z,1}^+}$, and $(\eta_{z,1}^-)^{t_{z,1}^-}$, except with $(\beta_j, u_j)_{j \in \mathbf{N}}$ replaced by $(\beta_{n+j}, u_{n+j})_{j \in \mathbf{N}}$. By combining everything, we get the statement of the lemma. \square

Lemma 6.24. *Let $z \in \mathbf{D}$, $m \in \mathbf{N}$, and $k \leq n \in \mathbf{N}$. Recall the definitions of the events from Section 6.3. We have*

$$\mathbf{P}(E_{0,n}^m(z)) \asymp \mathbf{E}(E_{0,k}^m(z)) \mathbf{P}(E_{0,n-k}^{m+k}(z)) \quad (6.33)$$

with proportionality constants independent of n , m , and k and uniform for $z \in B_d(0)$.

Proof. We have

$$\mathbf{P}(E_{0,n}^m(z)) = \mathbf{P}(E_{0,k}^m(z)) \mathbf{P}(E_{k,n}^m(z) | E_{0,k}^m(z)).$$

Furthermore

$$\mathbf{P}(E_{k,n}^m(z) | E_{0,k}^m(z)) = \mathbf{P}(L_{z,k+1} | E_{0,k}^m(z)) \mathbf{P}(E_{k,n}^m(z) | L_{z,k+1} \cap E_{0,k}^m(z)). \quad (6.34)$$

The first factor on the right in (6.34) is bounded below by a positive constant by Lemma 6.22. Conditional on $\mathcal{F}_{z,k+1}^0 \vee \mathcal{F}_{z,k}$, the event $E_{k,n}^m(z)$ is determined by $\eta_{z,k+1}$ and the auxiliary flow lines $(\eta_{z,j}^\pm)^{\widehat{t}_{z,j}^\pm}$ for $j \geq k+1$. Hence that Lemma 6.23 implies $\mathbf{P}(E_{k,n}^m(z) | L_{z,k+1} \cap E_{0,k}^m(z)) \asymp \mathbf{P}(E_{0,n-k}^{m+k}(z))$, as required. \square

Lemma 6.25. *For each $z \in B_d(0)$ and each $n, m \in \mathbf{N}$, we have*

$$e^{-\bar{\beta}_{m,m+n}\gamma^*(q)-3\gamma_0^*(q)\bar{u}_{m,m+n}} \leq \mathbf{P}(E_{0,n}^m(z)) \leq e^{-\bar{\beta}_{m,m+n}\gamma^*(q)+3\gamma_0^*(q)\bar{u}_{m,m+n}}$$

with the implicit constants independent of n and uniform for $z \in B_d(0)$.

Proof. Let C_*^{-1} and C_* be the proportionality constants in (6.33). By Lemma 6.24 we have

$$C_*^{-n} \prod_{j=0}^{n-1} \mathbf{P}(E_{z,1}^{m+j}) \leq \mathbf{P}(E_{0,n}^m(z)) \leq C_*^n \prod_{j=0}^{n-1} \mathbf{P}(E_{z,1}^{m+j}),$$

with the implicit constant independent of β , n , and m and uniform for z in compacts. By (6.13) we then have

$$e^{-\bar{\beta}_{m,m+n}\gamma^*(q)-\gamma_0^*(q)\bar{u}_{m,m+n}} C_*^{-n} \prod_{j=m}^{m+n-1} C_{u_j}^{-1} \leq \mathbf{P}(E_{0,n}^m(z)) \leq e^{-\bar{\beta}_{m,m+n}\gamma^*(q)-\gamma_0^*(q)\bar{u}_{m,m+n}} C_*^n \prod_{j=m}^{m+n-1} C_{u_j}. \quad (6.35)$$

By condition 6 in Lemma 6.14 we have $C_*^n \leq e^{\gamma_0^*(q)\bar{u}_{m,m+n}}$. By condition 5 in Lemma 6.14 we also have $\prod_{j=m}^{m+n-1} C_{u_j} \leq e^{\gamma_0^*(q)\bar{u}_{m,m+n}}$. By plugging these estimates into (6.35) we get the desired conclusion. \square

Lemma 6.26. *Let $z, w \in B_d(0)$. Let λ be as in assertion 2 of Lemma 6.15. Choose $k \in \mathbf{N}$ such that $e^{-\bar{\beta}_{k+1}-\lambda(k+1)} \leq |z-w| \leq e^{-\bar{\beta}_k-\lambda k}$. Then for any $n \in \mathbf{N}$ with $\bar{\beta}_n - \lambda n \geq \bar{\beta}_{k+1} + \lambda(k+2)$, we have*

$$\mathbf{P}(E_{k,n}(z) \cap E_{k,n}(w) | E_k(z) \cap E_k(w)) \leq e^{\bar{\beta}_k o_k(1)} \mathbf{P}(E_{0,n-k}^k(z)) \mathbf{P}(E_{0,n-k}^k(w))$$

with the implicit constants independent of n and k , the $o_k(1)$ independent of n , and both deterministic and uniform for $z, w \in B_d(0)$.

Proof. Throughout, we require implicit constants and $o_k(1)$ terms to satisfy the conditions of the statement of the lemma.

Let k' be the least integer such that $\bar{\beta}_{k'} - \lambda k' \geq \bar{\beta}_{k+1} + \lambda(k+2)$. Note $k' \leq n$. Let $\dot{P}_{z,k'}$ be the event that $\text{diam}(\hat{D}_{z,k'}) \leq \frac{1}{2}e^{-\bar{\beta}_{k'} + \lambda k'}$ and the event $\dot{K}_{z,k'}$ defined in (6.31) occurs. Define $\dot{P}_{w,k'}$ similarly. By Lemma 6.21, we have $\hat{D}_{z,k'} = \hat{D}_{z,k'}$ on $E_{k'}(z)$, so the objects involved in the definition (6.32) of $\dot{E}_{0,n-k'}^{k'}(z)$ agree with the objects involved in the definition of $E_{k',n}(z)$ on this event. Similar statements hold with w in place of z . By combining this observation with Condition 2 in Subsection 6.1 and assertion 4 of Lemma 6.15, we get that if \dot{a} (as defined just above (6.31)) is chosen sufficiently small, depending only on a , then

$$E_n(z) \subset \dot{E}_{n-k'}^{k'}(z) \cap \dot{P}_{z,k'}, \quad \text{and} \quad E_n(w) \subset \dot{E}_{n-k'}^{k'}(w) \cap \dot{P}_{w,k'}.$$

Therefore,

$$\begin{aligned} \mathbf{P}(E_{k,n}^k(z) \cap E_{k,n}^k(w) \mid E_k(z) \cap E_k(w)) &= \mathbf{P}(E_n(z) \cap E_n(w) \mid E_k(z) \cap E_k(w)) \\ &\leq \mathbf{P}(\dot{E}_{n-k'}^{k'}(z) \cap \dot{P}_{z,k'} \cap \dot{E}_{n-k'}^{k'}(w) \cap \dot{P}_{w,k'} \mid E_k(z) \cap E_k(w)) \\ &\leq \mathbf{P}(\dot{E}_{n-k'}^{k'}(z) \cap \dot{E}_{n-k'}^{k'}(w) \mid E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}) \end{aligned} \quad (6.36)$$

So, we need only estimate the last line of (6.36).

On the event $\dot{P}_{z,k'} \cap \dot{P}_{w,k'}$, the domains $\hat{D}_{z,k'}$ and $\hat{D}_{w,k'}$ are disjoint. By assertion 4 of Lemma 6.15, on the event $E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}$ we have $\hat{D}_{z,k'} \cup \hat{D}_{w,k'} \subset \hat{D}_{z,k} \cap \hat{D}_{w,k}$. The event $E_k(z)$ is determined by $\hat{D}_{z,k}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,k}}$. Consequently, the event $E_k(z) \cap \dot{P}_{z,k'}$ is determined by $\hat{D}_{z,k'}$ and $h|_{\mathbf{D} \setminus \hat{D}_{z,k'}}$. Similar statements hold with w in place of z .

Let \mathcal{H} be the σ -algebra generated by $\hat{D}_{z,k'}$, $\hat{D}_{w,k'}$, $h|_{\partial \hat{D}_{z,k'}}$, $h|_{\partial \hat{D}_{w,k'}}$, and \mathcal{F}_k . Note that $\dot{P}_{z,k'}$ and $\dot{P}_{w,k'}$ belong to \mathcal{H} (the boundary data of $h|_{\partial \hat{D}_{z,k'}}$ determines the locations of $\dot{x}_{z,n}$ and $\dot{y}_{z,n}$ and similarly with w in place of z). The above considerations together with Lemma 6.21 imply that the events $\dot{E}_{n-k'}^{k'}(z)$ and $\dot{E}_{n-k'}^{k'}(w)$ are conditionally independent given $E_k(z) \cap E_k(w)$ and \mathcal{H} on the event $\dot{P}_{z,k'} \cap \dot{P}_{w,k'}$. We thus have

$$\mathbf{P}(\dot{E}_{n-k'}^{k'}(z) \cap \dot{E}_{n-k'}^{k'}(w) \mid \mathcal{H}) \mathbf{1}_{E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}} = \mathbf{P}(\dot{E}_{n-k'}^{k'}(z) \mid \mathcal{H}) \mathbf{P}(\dot{E}_{n-k'}^{k'}(w) \mid \mathcal{H}) \mathbf{1}_{E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}}. \quad (6.37)$$

By Lemmas 6.22 and 6.23,

$$\mathbf{P}(\dot{E}_{n-k'}^{k'}(z) \mid \mathcal{H}) \mathbf{1}_{E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}} \asymp \mathbf{P}(E_{0,n-k'}^{k'}(z)) \mathbf{1}_{E_k(z) \cap E_k(w) \cap \dot{P}_{z,k'} \cap \dot{P}_{w,k'}} \quad (6.38)$$

and similarly with z and w interchanged. By (6.36), (6.37), and (6.38),

$$\mathbf{P}(E_{k,n}^k(z) \cap E_{k,n}^k(w) \mid E_k(z) \cap E_k(w)) \preceq \mathbf{P}(E_{0,n-k'}^{k'}(z)) \mathbf{P}(E_{0,n-k'}^{k'}(w)). \quad (6.39)$$

By Lemma 6.24 we have

$$\mathbf{P}(E_{0,n-k'}^{k'}(z)) \asymp \frac{\mathbf{P}(E_{0,n-k}^k(z))}{\mathbf{P}(E_{0,k'-k}^k(z))}. \quad (6.40)$$

By Lemma 6.25,

$$\mathbf{P}(E_{0,k'-k}^k(z)) \succeq e^{-\bar{\beta}_{k,k'} \gamma^*(q) - 3\gamma_0^*(q) \bar{u}_{k,k'}}.$$

By our choice of k' we have $\bar{\beta}_{k,k'} \leq \lambda(k+k'+1) + \beta_{k'}$. Since β_j is increasing in j we have $k' - k = o_k(1)\bar{\beta}_k$ and $\lambda(k+k'+1) = o_k(1)\bar{\beta}_k$. By condition 3 in Lemma 6.14 we have $\beta_{k'} \leq \beta_{k+o_k(1)} \leq \bar{\beta}_k o_k(1)$. Therefore $\mathbf{P}(E_{0,k'-k}^k(z)) = e^{k o_k(1)\bar{\beta}_k}$. Hence (6.40) implies

$$\mathbf{P}(E_{0,n-k'}^{k'}(z)) \preceq e^{\bar{\beta}_k o_k(1)} \mathbf{P}(E_{0,n-k}^k(z)).$$

A similar assertion holds with w in place of z . We conclude by combining this with (6.39). \square

Proof of Proposition 6.19. We have

$$\begin{aligned} \mathbf{P}(E_n(z) \cap E_n(w)) &= \mathbf{P}(E_{k,n}^k(z) \cap E_{k,n}^k(w) \cap E_k(z) \cap E_k(w)) \quad (\text{by definition}) \\ &\leq \mathbf{P}(E_{k,n}^k(z) \cap E_{k,n}^k(w) \mid E_k(z) \cap E_k(w)) \mathbf{P}(E_k(z)) \\ &\preceq e^{\bar{\beta}_k o_k(1)} \mathbf{P}(E_{0,n-k}^k(z)) \mathbf{P}(E_{0,n-k}^k(w)) \mathbf{P}(E_k(z)) \quad (\text{by Lemma 6.26}). \end{aligned}$$

By Lemma 6.24 we have

$$\mathbf{P}(E_{0,n-k}^k(w)) \asymp \frac{\mathbf{P}(E_n(w))}{\mathbf{P}(E_k(w))}, \quad \mathbf{P}(E_{0,n-k}^k(z)) \mathbf{P}(E_k(z)) \asymp \mathbf{P}(E_n(z))$$

By combining the above relations we get (6.29). \square

7 Lower bounds for multifractal and integral means spectra

7.1 Setup

Let η be a chordal SLE_κ from $-i$ to i in \mathbf{D} . Let D_η be as in Theorem 1.1 and define the sets $\tilde{\Theta}^s(D_\eta)$ and $\Theta^s(D_\eta)$ as in Section 1.1. The goal of this section is to obtain lower bounds on $\dim_{\mathcal{H}} \tilde{\Theta}^s(D_\eta)$ and $\dim_{\mathcal{H}} \Theta^s(D_\eta)$, and thereby complete the proof of Theorem 1.1. We accomplish this using the estimates of Section 6.

Throughout this section we use the notation defined in Sections 6.2 and 6.3, with

$$q = \frac{s}{1-s}.$$

We also continue to use the notation $\mathcal{B}_\beta = B_{e^{-\beta}}(0)$. In particular, we recall the definition (6.16) of the n -perfect points and the definition of the exponents $\gamma^*(q)$ and $\gamma_0^*(q)$ from (6.1). In the next two sections we will use the n -perfect points to define various notions of “perfect points” which are contained in the sets we are interested in and which will allow us to obtain lower bounds on their Hausdorff dimensions.

In order to prove that the perfect points are contained in our sets of interest, we will need the following technical lemma.

Lemma 7.1. *Let $\Psi_\eta : D_\eta \rightarrow \mathbf{D}$ be as in Section 4.1. Suppose $z \in \mathcal{P}_k \cap D_\eta$. For $n \leq k-1$ let $I_{z,n}$ be the image under Ψ_η of $\eta \cap \hat{D}_{z,n}$. If the parameters for the events of Section 6.2 are chosen appropriately, independently of n and of $z \in B_d(0)$, we have the following.*

1. We have $e^{-\bar{\beta}_n(q+1)-3\bar{u}_n} \preceq \text{length } I_{z,n} \preceq e^{-\bar{\beta}_n(q+1)+3\bar{u}_n}$.
2. If $n \leq k-2$ the distance from $I_{z,n+1}$ to $\partial I_{z,n}$ is proportional to the length of $I_{z,n}$.
3. If $x \in I_{z,n}$ then there exists $\delta_n > 0$ such that $|(\Psi_\eta^{-1})'((1-\delta_n)x)| \asymp \delta_n^{s+o_n(1)}$ and $\delta_n \asymp e^{-\bar{\beta}_n(q+1+o_n(1))}$.

The implicit constants are independent of n and both the $o_n(1)$ and the implicit constants are deterministic and independent of k , x , and $z \in B_d(0)$.

Proof. Fix n , k , and z as in the statement of the lemma. Throughout the proof we assume $E_k(z)$ occurs and require all constants (either referred to as such or implicit in \asymp , etc.) to be deterministic and independent of n , k , and $z \in B_d(0)$. See Figure 7.1 for an illustration of the argument.

Recall the pocket $\hat{D}_{z,n}$ formed by the auxiliary flow lines $\hat{\eta}_{z,n}^\pm$ from Section 6.2. The map $\hat{p}_{z,n} : \hat{D}_{z,n} \rightarrow \mathbf{D}$ defined in Section 6.2 takes z to 0 and $\eta \cap \hat{D}_{z,n}$ to the curve $\eta_{z,n+1}^0$, whose endpoints are $x_{z,n+1}$ and $y_{z,n+1}$. Note that condition 2 in the definition of $E_{\beta_n}^{q;u_n}(\cdot)$ implies lower bound on $|x_{z,n+1} - y_{z,n+1}|$, depending only on the parameter a .

By conditions 1 and 4 in the definition of $L_{z,n+1}$, there is a unique arc A^0 of $\partial \mathcal{B}_{\bar{\Delta}/2}$ which lies to the right of $\eta_{z,n+1}^0$ and disconnects $\eta_{z,n+1}^0 \cap B_{\bar{\Delta}}$ from $[x_{z,n+1}^*, y_{z,n+1}^*]_{\partial \mathbf{D}}$ in $\mathbf{D} \setminus \eta_{z,n+1}^0$ (c.f. Remark 6.10). Let

w^0 be the point of A^0 closest to the midpoint of $[x_{z,n+1}^*, y_{z,n+1}^*]_{\partial \mathbf{D}}$. Let D^0 be the connected component of $\mathbf{D} \setminus \eta_{z,n+1}^0$ containing $[x_{z,n+1}^*, y_{z,n+1}^*]_{\partial \mathbf{D}}$ on its boundary.

Observe that the harmonic measure from w^0 in D^0 of any sub-arc of $[x_{z,n+1}^*, y_{z,n+1}^*]_{\partial \mathbf{D}}$ is proportional to the length of that sub-arc. Furthermore, $\text{hm}^{w^0}(\eta_{z,n+1}^0; D^0) \asymp 1$. Define $\psi_{z,n}^F$ as in the definition of $F_{z,n}$. By condition 2 in the definition of $E_{\beta_n}^{q;u_n}$ and condition 2 in the definition of $E_{z,n}$, the arc of $\partial \mathbf{D}$ which is the image of the right side of $\eta_{z,n}^{T^*,n}$ (resp. the left side of $\bar{\eta}_{z,n}^{T^*,n}$) under $\psi_{z,n}^F$ has length $\asymp 1$. By conformal invariance of Brownian motion and condition 4 the definition of $F_{z,n}$, provided r is chosen sufficiently small relative to $\tilde{\Delta}$, the harmonic measure from $(\psi_{z,n}^F \circ p_{z,n}^{-1})(w^0)$ in the right connected component of $\mathbf{D} \setminus \psi_{z,n}^F(\eta_{z,n}^0)$ of each of these two sub-arcs is $\asymp 1$.

Let $w = \hat{p}_{z,n}^{-1}(w^0)$. It follows from the above considerations and conformal invariance of Brownian motion that (notation as in Section 2.1)

$$\text{hm}^w(\eta^{T^*,n}; D_\eta) \asymp \text{hm}^w(\bar{\eta}^{T^*,n}; D_\eta) \asymp 1 \quad \text{and} \quad \text{hm}^w(\eta \cap \hat{D}_{z,n}; D_\eta) \asymp 1. \quad (7.1)$$

By Lemma B.4 and condition 1 in the definition of $L_{z,1}$, we thus have

$$|\Psi'_\eta(w)| \asymp |\Phi'_{z,n}(w)|, \quad \text{and} \quad \text{dist}(w, \eta) \asymp \text{dist}\left(w, \eta^{T^*,n} \cup \bar{\eta}^{T^*,n}\right). \quad (7.2)$$

If $\tilde{\Delta}$ is chosen sufficiently large, independently of n and $z \in B_d(0)$, then by the Koebe growth theorem applied to $\hat{p}_{z,n}^{-1}$, we have $|w - z| \leq \frac{1}{2} \text{dist}(z, \eta^{T^*,n} \cup \bar{\eta}^{T^*,n})$. By the Koebe distortion theorem we then have $|\Phi'_{z,n}(w)| \asymp |\Phi'_{z,n}(z)|$, which by assertion 1 of Lemma 6.15 is bounded between constants times $e^{-\bar{\beta}_n(q+1)+2\bar{u}_n}$ and $e^{-\bar{\beta}_n(q+1)-2\bar{u}_n}$. Therefore,

$$e^{-\bar{\beta}_n q + 2\bar{u}_n} \preceq |\Psi'_\eta(w)| \preceq e^{-\bar{\beta}_n q - 2\bar{u}_n}. \quad (7.3)$$

Moreover, by assertion 2 of Lemma 6.15 and assertion 6 of Lemma 6.14, $\text{dist}(z, \eta^{T^*,n} \cup \bar{\eta}^{T^*,n})$ is bounded between constants times $e^{-\bar{\beta}_n(q+1)+\bar{u}_n}$ and $e^{-\bar{\beta}_n(q+1)-\bar{u}_n}$, so

$$e^{-\bar{\beta}_n - \bar{u}_n} \preceq \text{dist}(w, \eta^{T^*,n} \cup \bar{\eta}^{T^*,n}) \preceq e^{-\bar{\beta}_n + \bar{u}_n}. \quad (7.4)$$

Let $\tilde{w} = \Psi_\eta(w)$. By (7.3), (7.4), and the Koebe quarter theorem we have $e^{-\bar{\beta}_n(q+1)-3\bar{u}_n} \preceq 1 - |\tilde{w}| \preceq e^{-\bar{\beta}_n(q+1)+3\bar{u}_n}$. By (7.1) and conformal invariance of harmonic measure,

$$\text{dist}(\tilde{w}, I_{z,n}) \asymp \text{length}(I_{z,n}) \asymp 1 - |\tilde{w}|. \quad (7.5)$$

This proves assertion 1.

To prove assertion 2, we observe that the harmonic measure from w^0 , as defined above, of each of $(\eta_{z,n+1}^0)^{\sigma_{z,n+1}}$ and $(\bar{\eta}_{z,n+1}^0)^{\bar{\sigma}_{z,n+1}}$ is $\asymp 1$, where here $\sigma_{z,n+1}$ and $\bar{\sigma}_{z,n+1}$ are as in the definition of $L_{z,n}$. It therefore follows from conformal invariance of harmonic measure the distance from the endpoints of $I_{z,n}$ to the endpoints of $I_{z,n+1}$ is proportional to $1 - |\tilde{w}|$. We conclude by means of (7.5).

To complete the proof of assertion 3, suppose given $x \in I_{z,n}$. By (7.5) the angle between the tangent line to $\partial \mathbf{D}$ at x and the segment $[x, \tilde{w}]$ is $\asymp 1$. Hence we can find $\delta_n \asymp 1 - |\tilde{w}|$ and $r \in (0, 1)$ with $\log r \asymp 1$ such that $\tilde{w} \in B_{r\delta_n}((1 - \delta_n)x)$. By the Koebe distortion theorem we have $|(\Psi_\eta^{-1})'((1 - \delta_n)x)| \asymp |(\Psi_\eta^{-1})'(\tilde{w})|$. By combining this with 7.4 we conclude that assertion 3 holds. \square

7.2 Lower bound for the Hausdorff dimension of the subset of the curve

In this subsection we will prove a lower bound on the Hausdorff dimension of the sets $\Theta^s(D_\eta)$. Assume we are in the setting of Section 7.1.

We define the set \mathcal{P} of *perfect points* as follows. Let λ be the constant from Proposition 6.19. For $n \in \mathbb{N}$, let n' be the greatest integer such that $\bar{\beta}_n - \lambda n \geq \bar{\beta}_{n'+1} + \lambda(n' + 2)$. Let

$$\epsilon_n := e^{-\bar{\beta}_{n'+1} - \lambda(n'+2)}. \quad (7.6)$$

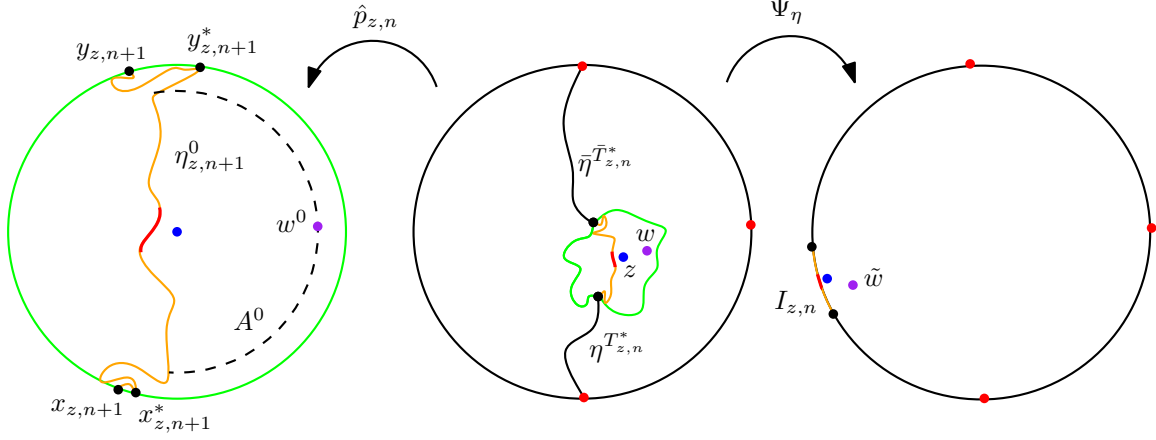


Figure 7.1: An illustration of the proof of Lemma 7.1. We use harmonic measure from w and its images under various maps to deduce the desired conditions on the orange arc $I_{z,n}$ from the definitions of the events of Section 6.2. Also shown (in red) is the arc $I_{z,n+1}$ appearing in assertion 2 and its images under the various maps.

Note that Lemma 6.14 implies $e^{-\bar{\beta}_n} = \epsilon_n^{1+o_n(1)}$. Our reason for choosing this value of ϵ_n is that the pockets $\hat{D}_{z,n}$ and $\hat{D}_{w,n}$ are disjoint on $E_n(z) \cap E_n(w)$ provided $|z - w| \geq \epsilon_n$ (see Lemma 6.15).

Choose a collection \mathcal{C}_n of $\asymp \epsilon_n^{-2}$ points in $B_d(0)$, no two of which lie within distance ϵ_n of each other. For $z \in \mathcal{C}_n$ let $B^n(z)$ be the ball of radius ϵ_n centered at z . Let $\mathcal{C}'_n = \mathcal{C}_n \cap \mathcal{P}_n$ be the set of $z \in \mathcal{C}_n$ for which $E_n(z)$ occurs. Let

$$\mathcal{P} := \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \bigcup_{z \in \mathcal{C}'_k} B^k(z)}. \quad (7.7)$$

Lemma 7.2. *Define \mathcal{P} as in (7.7). We have $\mathcal{P} \subset \Theta^s(D_\eta)$ for $s = q/(q+1)$. In fact, if $w \in \mathcal{P}$, then for $\epsilon > 0$ we have*

$$|(\Psi_\eta^{-1})'((1-\epsilon)\Psi_\eta(w))| \asymp \epsilon^{-s+o_\epsilon(1)}, \quad (7.8)$$

with the implicit constants and the $o_n(1)$ deterministic and uniform for $w \in B_d(0)$.

Proof. Fix $w \in \mathcal{P}$. Since η is closed, it is clear that $w \in \eta$. It remains to prove (7.8). By definition of \mathcal{P} , if we are given $n \in \mathbf{N}$, then we can find $k \geq n+1$ and $z \in \mathcal{C}'_k$ such that $|z - w| \leq e^{-2\bar{\beta}_{n+1}}$. By Lemma 6.15 we have $w \in \hat{D}_{z,n}$, so $\Psi_\eta(w) \in I_{z,n}$, as defined in Lemma 7.1. Let δ_n be as in that lemma with $x = \Psi_\eta(w)$.

By the Koebe distortion theorem, for $\epsilon \in [\delta_{n+1}, \delta_n]$ we have

$$\frac{1 - (\delta_n - \delta_{n+1})/\delta_n}{(1 + (\delta_n - \delta_{n+1})/\delta_n)^3} \leq \frac{|(\Psi_\eta^{-1})'((1-\epsilon)\Psi_\eta(w))|}{|(\Psi_\eta^{-1})'((1-\delta_n)\Psi_\eta(w))|} \leq \frac{1 + (\delta_n - \delta_{n+1})/\delta_n}{(1 - (\delta_n - \delta_{n+1})/\delta_n)^3}. \quad (7.9)$$

By Lemma 7.1 we have

$$1 - (\delta_n - \delta_{n+1})/\delta_n \asymp e^{-\beta_n + \bar{\beta}_n o_n(1)},$$

which is proportional to $e^{\bar{\beta}_n o_n(1)} \asymp \epsilon^{o_\epsilon(1)}$ by Lemma 6.14. We furthermore have $\delta_n = \epsilon^{1+o_\epsilon(1)}$. Hence (7.9) and Lemma 7.1 imply $|(\Psi_\eta^{-1})'((1-\epsilon)\Psi_\eta(w))| \asymp \epsilon^{-s+o_\epsilon(1)}$, as required. \square

Proposition 7.3. *Let s_-, s_+ be as in Theorem 1.1. For each $s \in (s_-, s_+)$, we a.s. have*

$$\dim_{\mathcal{H}} \Theta^s(D_\eta) \geq \xi(s),$$

where $\xi(s)$ is as in (1.4).

Proof. For a Borel measure ν on a metric space X and $\alpha > 0$, write

$$I_\alpha(\nu) = \int_X \int_X \frac{d\nu(z) d\nu(w)}{|z - w|^\alpha} \quad (7.10)$$

for the α -energy of ν . By standard results for Hausdorff dimension (see [MP10, Theorem 4.27]) a metric space which admits a positive finite measure with finite α -energy has Hausdorff dimension at least α . In view of Lemma 7.2, we are lead to construct such a measure ν on \mathcal{P} for each $\alpha < \xi(s)$. We do this via the usual argument (see, e.g. [MW14, HMP10, Bef08]).

Define the events $E_n(z)$ as in Section 6.3 and the sets of points \mathcal{C}_n and \mathcal{C}'_n and the balls $B^n(z)$ as in the definition of \mathcal{P} (right above (7.7)). Let ϵ_n be as in (7.6).

For each $n \in \mathbf{N}$, define a measure ν_n on \mathbf{D} by

$$d\nu_n(x) = \sum_{z \in \mathcal{C}_n} \frac{\mathbf{1}_{E_n(z)}}{\mathbf{P}(E_n(z))} \mathbf{1}_{(x \in B^n(z))} dx.$$

Then $\mathbf{E}(\nu_n(\mathbf{D})) \asymp 1$. Moreover,

$$\mathbf{E}(\nu_n(\mathbf{D})^2) \preceq \epsilon_n^4 \sum_{z \neq w \in \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} + \epsilon_n^4 \sum_{z \in \mathcal{C}_n} \frac{1}{\mathbf{P}(E_n(z))}.$$

By Lemma 6.25 and Proposition 6.19, this is bounded by an n -independent constant times

$$\epsilon_n^4 \sum_{z \neq w \in \mathcal{C}_n} |z - w|^{-\gamma^*(q) + o_{|z-w|}(1) + o_n(1)} + \epsilon_n^4 \sum_{z \in \mathcal{C}_n} \epsilon_n^{-\gamma^*(q) + o_n(1)},$$

with the $o_{|z-w|}(1)$ independent of n . For $s \in (s_-, s_+)$ we have $\gamma^*(q) = \gamma(s)/(1-s) < 2$. Therefore, for sufficiently large n , $\mathbf{E}(\nu_n(\mathbf{D})^2)$ is bounded above by a finite, n -independent constant. By the Vitalli convergence theorem, we can a.s. find a subsequence of the measures ν_n which converges weakly to a measure ν whose total mass is bounded above by some deterministic constant, and whose expected mass is positive.

On the other hand, we have

$$\begin{aligned} \mathbf{E}(I_\alpha(\nu_n)) &= \sum_{z, w \in \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} \iint_{B^n(z) \times B^n(w)} \frac{1}{|x - y|^\alpha} dx dy \\ &= \sum_{z \neq w \in \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} \iint_{B^n(z) \times B^n(w)} \frac{1}{|x - y|^\alpha} dx dy \\ &\quad + \sum_{z \in \mathcal{C}_n} \frac{1}{\mathbf{P}(E_n(z))} \iint_{B^n(z) \times B^n(z)} \frac{1}{|x - y|^\alpha} dx dy \\ &\preceq \sum_{z \neq w \in \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} \frac{\epsilon_n^4}{|z - w|^\alpha} + \sum_{z \in \mathcal{C}_n} \frac{\epsilon_n^{4-\alpha}}{\mathbf{P}(E_n(z))} \\ &\preceq \sum_{z \neq w \in \mathcal{C}_n} |z - w|^{-\gamma^*(q) - \alpha + o_{|z-w|}(1) + o_n(1)} \epsilon_n^4 + \epsilon_n^{2-\alpha - \gamma^*(q) + o_n(1)}. \end{aligned}$$

We have $\gamma^*(q) + \alpha < 2$ for $s \in (s_-, s_+)$ and $\alpha < \xi(s)$, so the above expression is $\preceq 1$. We conclude that with positive probability, there exists a weak subsequential limit ν of the measures (ν_n) supported on \mathcal{P} and satisfying $\nu(\mathcal{P}) > 0$ and $I_\alpha(\nu) < \infty$. Hence [MP10, Theorem 4.27] and Lemma 7.2 imply that with positive probability, we have $\dim_{\mathcal{H}} \Theta^s(D_\eta) \geq \xi(s)$. Proposition 2.16 implies that this in fact a.s. holds. \square

7.3 Lower bound for the Hausdorff dimension of the subset of the circle

In order to get a lower bound on the Hausdorff dimension of $\tilde{\Theta}^s(D_\eta)$, we will need a different set of perfect points. Define ϵ_n , the sets \mathcal{C}_n , \mathcal{C}'_n as in the definition (7.7) of \mathcal{P} . For $z \in \mathcal{C}'_n$, let $I_{z,n-1}$ be as in the statement

of Lemma 7.1. Let $x_{z,n}$ be the midpoint of $I_{z,n-1}$ and let $I'_{z,n}$ be the arc of length ϵ_n^{q+1} centered at $x_{z,n}$. By Lemma 7.1 we have $\text{length}(I'_{z,n}) = \text{length}(I_{z,n-1})^{1+o_n(1)}$. Let

$$\tilde{\mathcal{P}} := \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \bigcup_{z \in \mathcal{C}'_k} I'_{z,k-1}}. \quad (7.11)$$

Lemma 7.4. *Define $\tilde{\mathcal{P}}$ as in (7.11). If $\tilde{\Delta}$ and β_1 are chosen sufficiently large then $\tilde{\mathcal{P}} \subset \tilde{\Theta}^s(D_\eta)$ for $s = q/(q+1)$. In fact, if $x \in \tilde{\mathcal{P}}$, then for $\epsilon > 0$ we have*

$$|(\Psi_\eta^{-1})'((1-\epsilon)x)| \asymp \epsilon^{-s+o_\epsilon(1)},$$

with the implicit constants and the $o_\epsilon(1)$ deterministic and uniform in x .

Proof. If $x \in \tilde{\mathcal{P}}$ then for any given $n \in \mathbf{N}$ we can find $k \geq n$ and $z \in \mathcal{C}'_k$ such that x lies within distance $\text{length}(I'_{z,n})^2$ of $I'_{z,k}$. If k is chosen sufficiently large, depending on n , then by assertions 1 and 2 of Lemma 7.1 we have $x \in I_{z,n}$. We then conclude as in the proof of Lemma 7.2. \square

Lemma 7.5. *For each n there is an integer $m_n \leq n$ such that the following is true. We have $\bar{\beta}_n - \bar{\beta}_{m_n} = \bar{\beta}_n o_n(1)$ and if $z, w \in \mathcal{C}'_n$ with $|z - w| \geq e^{-\bar{\beta}_{m_n}+1}$ then we have $\text{dist}(I'_{z,n}, I'_{w,n}) \geq |z - w|^{q+1+o_{|z-w|}(1)}$, with the $o_{|z-w|}(1)$ and implicit constants deterministic, independent of n , and uniform for $z, w \in \mathcal{C}'_n$.*

Proof. We argue as in the proof of Lemma 6.26. Choose $k \in \mathbf{N}$ such that $e^{-\beta_{k+1}-\lambda(k+1)} \leq |z - w| \leq e^{-\beta_k-\lambda k}$. Let k' be the least integer such that $\bar{\beta}_{k'} - \lambda k' \geq \bar{\beta}_{k+1} + \lambda(k+1)$. By our choice (7.6) of ϵ_n we have $k' \leq n-1$. By Lemma 6.15, $\hat{D}_{z,k'} \cap \hat{D}_{w,k'} = \emptyset$ and hence $I_{z,k'} \cap I_{w,k'} = \emptyset$. If $\text{length}(I'_{z,n}) \leq \text{length}(I_{z,k'+1})$ then by assertions 1 and 2 of Lemma 7.1 we have

$$\text{dist}(x_{z,n}, x_{w,n}) \succeq e^{-\bar{\beta}_{k'+1}(q+1)-3\bar{u}_{k'+1}} \succeq |z - w|^{q+1+o_{|z-w|}(1)}.$$

On the other hand, by assertion 1 of Lemma 7.1 we have $\text{length}(I'_{z,n}) \leq \text{length}(I_{z,k'+1})$ provided $\bar{\beta}_{k'+1}(q+1) + 3\bar{u}_{k'+1} \leq (\bar{\beta}_n - \lambda n + \bar{\beta}_n o_n(1))(q+1)$, or equivalently provided

$$\bar{\beta}_n - \bar{\beta}_{k'+1} \geq \frac{3\bar{u}_{k'+1} + \lambda n + \bar{\beta}_n o_n(1)}{q+1}.$$

It follows from Lemma 6.14 that we can choose $m_n \leq n$ such that $\bar{\beta}_n - \bar{\beta}_{m_n} = \bar{\beta}_n o_n(1)$ and $\text{length}(I'_{z,n}) \leq \text{length}(I_{z,k'+1})$ whenever $k' \leq m_n$. \square

Proposition 7.6. *Let s_-, s_+ be as in Theorem 1.1. For each $s \in (s_-, s_+)$ we a.s. have*

$$\dim_{\mathcal{H}} \tilde{\Theta}^s(D_\eta) \geq \tilde{\xi}(s),$$

where $\tilde{\xi}(s)$ is as in (1.3).

Proof. We argue as in the proof of Proposition 7.3. In particular, for any given $\alpha < \tilde{\xi}(s)$, we will construct a positive finite measure $\tilde{\nu}$ on $\tilde{\mathcal{P}}$ (as defined in (7.11) with finite α -energy, as defined in (7.10)).

Define ϵ_n as in (7.6). We require all implicit constants and $o_{|z-w|}(1)$ terms to be independent of n and uniform for $z, w \in \mathcal{C}_n$. For $n \in \mathbf{N}$, define a measure $\tilde{\nu}_n$ on $\partial \mathbf{D}$ by

$$d\tilde{\nu}_n(x) = \epsilon_n^{1-q} \sum_{z \in \mathcal{C}'_n} \frac{\mathbf{1}_{E_n(z)}}{\mathbf{P}(E_n(z))} \mathbf{1}_{(x \in I'_{z,k})} dx.$$

Then we have $\mathbf{E}(\tilde{\nu}_n(\partial \mathbf{D})) \asymp 1$.

As in the proof of Proposition 7.3 we have

$$\mathbf{E}(\tilde{\nu}_n(\partial \mathbf{D})^2) \preceq \epsilon_n^4 \sum_{z \neq w \in \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} + \epsilon_n^4 \sum_{z \in \mathcal{C}_n} \epsilon_n^{-\gamma^*(q)+o_n(1)} \preceq 1.$$

Let m_n be as in Lemma 7.5 and let \mathcal{K}_n be the set of pairs $(z, w) \in \mathcal{C}_n \times \mathcal{C}_n$ with $|z - w| \leq e^{-\bar{\beta}_{m_n}}$ and $z \neq w$. By Lemma 7.5 we have $\#\mathcal{K}_n \leq \epsilon_n^{2-o_n(1)}$.

By Lemma 6.25, Proposition 6.19, and Lemma 7.5, we have

$$\begin{aligned} \mathbf{E}(I_\alpha(\tilde{\nu}_n)) &= \epsilon_n^{2-2q} \sum_{(z,w) \in \mathcal{C}_n \times \mathcal{C}_n} \frac{\mathbf{P}(E_n(z) \cap E_n(w))}{\mathbf{P}(E_n(z))\mathbf{P}(E_n(w))} \iint_{I_{z,k} \times I_{w,k}} \frac{1}{|x-y|^\alpha} dx dy \\ &\preceq \sum_{(z,w) \notin \mathcal{K}_n, z \neq w} |z-w|^{-\gamma^*(q)+o_{|z-w|}(1)} |x_{z,n} - x_{w,n}|^{-\alpha} \epsilon_n^{2(1+q)+2-2q} \\ &\quad + \sum_{(z,w) \in \mathcal{K}_n} |z-w|^{-\gamma^*(q)+o_{|z-w|}(1)} \epsilon_n^{(2-\alpha)(q+1)+2-2q+o_n(1)} \\ &\quad + \sum_{z \in \mathcal{C}_n} \epsilon_n^{(2-\alpha)(q+1)+2-2q-\gamma^*(q)+o_n(1)} \\ &\preceq \epsilon_n^4 \sum_{z \neq w \in \mathcal{C}'_n} |z-w|^{-\gamma^*(q)-\alpha(q+1)+o_{|z-w|}(1)} + \epsilon_n^{(2-\alpha)(q+1)-2q-\gamma^*(q)+o_n(1)} + \epsilon_n^{(2-\alpha)(q+1)-2q-\gamma^*(q)+o_n(1)}. \end{aligned}$$

Note that for the middle term we used $|z-w| \geq \epsilon_n$ and $\#\mathcal{K}_n \leq \epsilon_n^{-2-o_n(1)}$. If $s \in (s_-, s_+)$ and $q = s/(1-s)$ we have $\gamma^*(q) + \alpha(q+1) < 2$ and $(2-\alpha)(1+q) - 2q - \gamma^*(q) > 0$ for $\alpha < \tilde{\xi}(s)$. It follows that we can a.s. find a subsequence of the measures $(\tilde{\nu}_n)$ which converges weakly a.s. to a finite positive limiting measure supported on $\tilde{\mathcal{P}}$ with finite α -energy. We then conclude using [MP10, Theorem 4.27], Lemma 7.4, and Proposition 2.16. \square

7.4 Proof of Theorem 1.1

This follows by combining Propositions 5.1, 5.6, 7.3, and 7.6. \square

Remark 7.7. In the case $\kappa = 4$, we have $s_+ = 1$, so the sets $\Theta^1(D_\eta)$ and $\tilde{\Theta}^1(D_\eta)$ for $\kappa = 4$ can be non-empty. We do not explicitly mention these sets in Theorem 1.1 because our results do not apply in full in this case. However, we do prove something about these sets. In particular, we prove in Proposition 5.1 that a.s. $\dim_{\mathcal{H}} \tilde{\Theta}^1(D_\eta) = 0$. Since $\dim_{\mathcal{H}}(\eta) = 3/2$ for $\kappa = 4$, we get a trivial upper bound of $3/2$ for $\dim_{\mathcal{H}} \Theta^1(D_\eta)$ in the case $\kappa = 4$. We do not prove a lower bound for $\dim_{\mathcal{H}} \Theta^1(D_\eta)$ in this paper, and we are not sure if the upper bound of $3/2$ is optimal.

7.5 Lower bound for the integral means spectrum

In this subsection we prove our lower bound for the bulk integral means spectrum of the SLE curve and thereby complete the proof of Corollary 1.8.

Proof of Corollary 1.8. Throughout, we consider a fixed realization and allow implicit constants to be random (but independent of the parameters of interest).

Fix $s \in [s_-, s_+]$ (as defined in 1.5 and 1.6) to be chosen later, and let $\tilde{\mathcal{P}}$ be the set of perfect points defined in (7.11). Also fix $\alpha < \tilde{\xi}(s)$. By the proof of Proposition 7.6, the probability of the event

$$E := \{\dim_{\mathcal{H}} \tilde{\mathcal{P}} > \alpha\}$$

is positive. Moreover, it is clear from the definition that we have $\tilde{\mathcal{P}} \subset \Psi_\eta^{-1}(\eta \cap B_d(0))$.

For $n \in \mathbf{N}$ let $\epsilon_n := 2^{-n}$. Let \mathcal{I}_n be the collection of arcs $[e^{2\pi i(k-1)\epsilon_n}, e^{2\pi i k \epsilon_n}]_{\partial \mathbf{D}}$ for $k \in \{1, \dots, 2^n\}$. Let \mathcal{I}'_n be the set of those arcs $I \in \mathcal{I}_n$ which intersect $\tilde{\mathcal{P}}$. Then \mathcal{I}'_n is a cover of $\tilde{\mathcal{P}}$ consisting of sets of diameter $\leq O_n(\epsilon_n)$. Hence on E we have

$$(\#\mathcal{I}'_n) \epsilon_n^\alpha \geq \mathcal{H}^\alpha(\tilde{\mathcal{P}}) \geq 1,$$

so

$$\#\mathcal{I}'_n \geq \epsilon_n^{-\alpha}.$$

For $I \in \mathcal{I}'_n$ choose $x_I \in I \cap \tilde{\mathcal{P}}$. Let $z_I = (1 - \epsilon_n)x_I$. By Lemma 7.4 we have $|(\Psi_\eta^{-1})'(z_I)| \succeq \epsilon_n^{-s+o_n(1)}$, with the $o_n(1)$ and the implicit constant independent of the choice of I and x_I .

Let J_I be the intersection of $(1 - \epsilon_n)I$ with the arc of $\partial B_{1-\epsilon_n}(0)$ centered at z_I of length $\epsilon_n^{1+\delta_n}$, where (δ_n) is a sequence of positive numbers with $\delta_n \rightarrow 0$ slower than the $o_n(1)$ above. Then the arcs J_I are disjoint for sufficiently large n and by the Koebe distortion theorem we have $|(\Psi_\eta^{-1})'(w)| \succeq \epsilon_n^{s+o_n(1)}$ for each $w \in J_I$. Each point of $\tilde{\mathcal{P}}$ is mapped into $B_{1-d/2}(0)$ by Ψ_η^{-1} . Hence for sufficiently large n and sufficiently small ζ (random), we have $J_I \subset A_{\epsilon_n}^\zeta(\Psi_\eta^{-1})$ for each $I \in \mathcal{I}'_n$, with $A_{\epsilon_n}^\zeta(\Psi_\eta^{-1})$ defined just below (1.10) with $\phi = \Psi_\eta^{-1}$. Hence on E , it holds for $a \in \mathbf{R}$ that

$$\int_{A_{\epsilon_n}^\zeta(\Psi_\eta^{-1})} |(\Psi_\eta^{-1})'(w)|^a dw \succeq \sum_{I \in \mathcal{I}'_n} \int_{J_I} |(\Psi_\eta^{-1})'(w)|^a dw \succeq \epsilon_n^{-\alpha-as+1+o_n(1)}.$$

Therefore, for any $a \in \mathbf{R}$, on E it holds that

$$\limsup_{n \rightarrow \infty} \frac{\log \int_{A_{\epsilon_n}^\zeta(\Psi_\eta^{-1})} |(\Psi_\eta^{-1})'(w)|^a dw}{\log \epsilon_n^{-1}} \geq \alpha + as - 1.$$

Thus $\text{IMS}_{D_\eta}^{\text{bulk}}(a) \geq \alpha + as - 1$ with positive probability.

By Proposition 2.17, this lower bound in fact holds a.s. Since $\alpha < \tilde{\xi}(s)$ is arbitrary, it follows that a.s.

$$\text{IMS}_{D_\eta}^{\text{bulk}}(a) \geq \tilde{\xi}(s) + as - 1 \quad (7.12)$$

In the notation of Corollary 1.8, this quantity is maximized over all $s \in [s_-, s_+]$ by taking $s = s_*(a)$ if $a \in [a_-, a_+]$; $s = s_-$ if $a < a_-$; and $s = s_+$ if $a > a_+$. Choosing this value of s in (7.12) gives us that the lower bound in (1.14) holds with a.s. for each fixed $a \in \mathbf{R}$ in the case $\kappa \leq 4$, $\underline{\rho} = 0$, and $V = D_\eta$.

By Proposition 2.17, this lower bound in fact holds a.s. for each choice of $\kappa > 0$, vector of weights $\underline{\rho}$, $t > 0$, and complementary connected component V of $\eta([0, t])$. By combining this with Proposition 5.7, we get that (1.14) holds a.s. for each fixed $a \in \mathbf{R}$ for each choice of $\kappa > 0$, vector of weights $\underline{\rho}$, $t > 0$, and complementary connected component V of $\eta([0, t])$. By Hölder's inequality, it follows the bulk integral means spectrum is a convex, hence continuous, function of a (c.f. [Mak98, Theorem 5.2] for a related, but much stronger, statement for the ordinary integral means spectrum). It follows that in fact 1.14 holds a.s. for all $a \in \mathbf{R}$ simultaneously. \square

A Proof of Proposition 3.10

In this appendix we will prove Proposition 3.10, which is one of the ingredients in the proof of Theorem 3.1. The proof will be completed in two stages. First, we will show that we can move the real part of the force point from $\text{Re } z$ to 0 without any pathological behavior (Lemma A.1). Then, we will use a forward/reverse SLE symmetry argument to rule out pathological behavior after the real part of the force point has first reached 0. See Figure A.1 for an illustration.

We adopt the following notation. Fix $z \in \mathbf{H}$ with $|\text{Re } z| \leq R$ and $\text{Im } z = \epsilon$. Let

$$Z_t = g_t(z) = X_t + iY_t. \quad (\text{A.1})$$

By (3.7), we have that under \mathbf{P}_*^z ,

$$dX_t = (\rho - 2) \frac{X_t}{|Z_t|^2} dt - \sqrt{\kappa} dB_t^z, \quad dY_t = \frac{2Y_t}{|Z_t|^2} dt, \quad X_0 = \text{Re } z, \quad Y_0 = \epsilon \quad (\text{A.2})$$

for B_t^z a \mathbf{P}_*^z -Brownian motion. Also let

$$S_0 := \inf\{t \geq 0 : X_t = 0\}. \quad (\text{A.3})$$

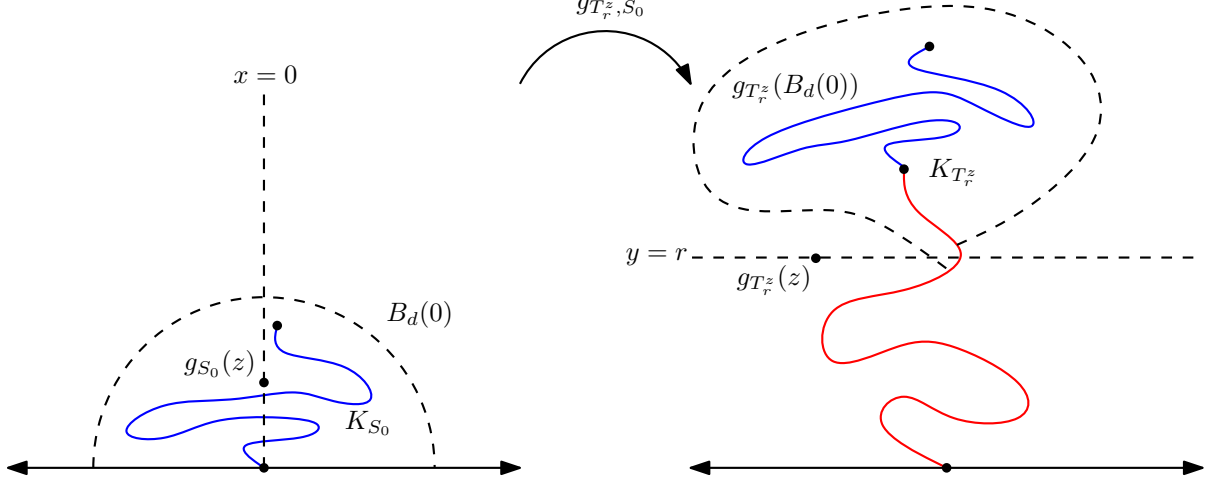


Figure A.1: An illustration of the proof of Proposition 3.10. First, we run the reverse Loewner flow with a force point at z until the first time S_0 that z is mapped to a point on the imaginary axis. We show in Section A.1 that for each $\zeta > 0$, it holds with uniformly positive probability (independent of the particular choice of z) that $S_0 \leq \zeta$, $Y_{S_0} = \text{Im } g_{S_0}(z) \leq 5\zeta^{1/2}$, and $K_{S_0} \subset B_d(0)$ for some $d > 0$ independent of the particular choice of z . Once we condition on the reverse Loewner flow up until time S_0 , the law of the maps $g_{S_0, v+S_0}$ which satisfy $g_{S_0, v+S_0} \circ g_{S_0} = g_{v+S_0}$ for $v \geq 0$ is that of a reverse $\text{SLE}_\kappa(\rho)$ Loewner flow with force point at $Z_{S_0} = g_{S_0}(z)$. In Section A.2, we show that the first time that the force point for such a Loewner flow reaches the line $\{\text{Im } w = r\}$ (i.e., $T_r^z - S_0$) is bounded independently of Z_{S_0} with high probability. Furthermore, the conformal map g_{S_0, T_r^z} is likely to “push” $B_d(0) \cap \mathbf{H}$ (and hence also K_{S_0}) away from the real axis; and the hull of this map (shown in red) is unlikely to be too large. These latter conditions together with Lemma 2.4 imply that $G(g_{T_r^z}^{-1}, \mu)$ occurs with uniformly positive probability for appropriate choice of μ .

A.1 Pushing the force point to the imaginary axis

In this subsection we will prove the following lemma, which deals with the setup on the left side in Figure A.1.

Lemma A.1. *Suppose we are in the setting of Proposition 3.10. Let $Z_t = X_t + iY_t$ be as in (A.1) and let S_0 be as in (A.3). Let $\zeta \in (0, 1)$. There exists $d > 0$ and $p_0 > 0$, independent of ϵ and of $X_0 \in [-R, R]$, such that whenever $\epsilon \leq \zeta$ the probability of the event $E_0 = E_0(\zeta, d)$ that*

1. $S_0 \leq \zeta$;
2. $Y_{S_0} \leq 5\zeta^{1/2}$;
3. $K_{S_0} \subset B_d(0)$;

is at least p_0 .

Proof. By symmetry we can assume without loss of generality that $X_0 > 0$. We will treat the conditions in the definition of E_0 in order.

Condition 1: Let

$$\nu > 1 \wedge \left(\frac{2(\rho - 2)}{\kappa} + 1 \right). \quad (\text{A.4})$$

Let \tilde{X} be $\sqrt{\kappa}$ times a Bessel process driven by $-B_t^z$, started from X_0 , of dimension ν . We have

$$d(\tilde{X}_t - X_t) = \frac{X_t(aX_t - (\rho - 2)\tilde{X}_t) + aY_t^2}{(X_t^2 + Y_t^2)\tilde{X}_t} dt,$$

where $a = \kappa(\nu - 1)/2 > 0 \wedge (\rho - 2)$. This is strictly positive whenever $X_t > \tilde{X}_t$ (since $\tilde{X}_t \geq 0$). This implies that a.s.

$$\tilde{X}_t \geq X_t, \quad \forall t \leq S_0. \quad (\text{A.5})$$

Our choice (3.9) for ρ implies that (A.4) holds for some Bessel dimension $\nu \in (0, 2)$, in which case \tilde{X} hits 0 before time ζ with uniformly positive probability [Law05, Proposition 1.21]. From this and (A.5) we conclude that we can find $p_0 > 0$ independent of ϵ and of $X_0 \in [-R, R]$ such that

$$\mathbf{P}(S_0 \leq \zeta) \geq 2p_0. \quad (\text{A.6})$$

Condition 2: By (A.2) we have that Y is increasing and $\partial_t Y_t^2 \leq 4$. Hence $Y_t \leq 4t^{1/2} + \epsilon$, so on the event $\{S_0 \leq \zeta\}$ we have $Y_{S_0} \leq 5\zeta^{1/2}$.

Condition 3: Let \tilde{X} be a Bessel process of dimension ν started from X_0 as in the proof of Condition 1. Since \tilde{X} and B^z are a.s. bounded up to time ζ and their laws do not depend on ϵ , it follows from (A.5) and (A.6) that we can find $C_0 > 0$, independent of ϵ and uniform for $X_0 \in [-R, R]$ such that the probability of the event E_0^* that

1. $S_0 \leq \zeta$;
2. $Y_{S_0} \leq 5\zeta^{1/2}$;
3. $\sup_{t \leq \zeta} |\sqrt{\kappa} B_t^z| \leq C_0$;
4. $\sup_{t \leq \zeta} |X_t| \leq C_0$;

is at least p_0 . Note that for the last condition we use (A.5).

By (A.2), we have for $t \leq S_0$ that

$$|\rho - 2| \int_0^t \frac{X_v}{X_v^2 + Y_v^2} dv \leq |X_0| + |X_t| + |\sqrt{\kappa} B_t^z|. \quad (\text{A.7})$$

In the case $\rho \neq 2$, it follows from (A.7) that on the event E_0^* ,

$$\int_0^t \frac{X_v}{X_v^2 + Y_v^2} dv \leq C_1 := \frac{R + 2C_0}{|\rho - 2|}. \quad (\text{A.8})$$

In the case $\rho = 2$, it follows from (A.2) that X is a constant times a Brownian motion, so in this case we can (using condition 1) find a possibly larger constant C_1 , still independent of ϵ , such that (A.8) holds with probability at least $1 - p_0/2$. In this case we add this latter condition to the event E_0^* .

Now consider some $b \in \mathbf{R}$, $|b| > 1$. Let $\delta > 0$. Let τ_b be the first time t that $|g_t(b)| \leq \delta$. By (3.7) and the Loewner equation, we have

$$g_t(b) = - \int_0^t \frac{2}{g_v(b)} dv + \rho \int_0^t \frac{X_v}{X_v^2 + Y_v^2} dv - \sqrt{\kappa} B_t^z + b.$$

So, it follows from (A.8) that on E_0^* we have

$$\inf_{t \leq S_0 \wedge \tau_b} |g_t(b)| \geq |b| - C_2,$$

where

$$C_2 = 2\zeta\delta^{-1} + |\rho|C_1 + C_0.$$

Hence if we take $|b| > 2C_2$, then we have $\inf_{t \leq S_0 \wedge \tau_b} |g_t(b)| \geq C_2$, which implies $\tau_b > S_0$ (provided we choose $\delta < C_2/2$).

In particular, if $b > 1$ is chosen sufficiently large (independent of ϵ and $X_0 \in [-R, R]$), then $g_{S_0}(-b)$ and $g_{S_0}(b)$ lie in \mathbf{R} . Therefore the map g_τ^{-1} takes ∂K_τ into $[-b, b]$. This implies that the harmonic measure from ∞ of K_τ in $\mathbf{H} \setminus K_\tau$ is at most $2\pi b$, so by [Law05, Equation 3.14], it follows that $\text{diam } K_{S_0}$ is bounded by a constant independent of ϵ and $X_0 \in [-R, R]$ on E_0^* . Since $\mathbf{P}(E_0^*) \geq p_0$, the lemma follows. \square

A.2 Pushing the force point starting from the imaginary axis

In light of the strong Markov property and Lemma A.1, we now need to consider the behavior of the process (A.2) if we start (X_0, Y_0) from $(0, y)$ for $y \in [\epsilon, 5\zeta^{1/2}]$ and ζ as in Lemma A.1. For this, we first need to review some calculations from [DMS14, Section 3]. Throughout this subsection, we assume $X_0 = 0$, $Y_0 = y \in [\epsilon, 5\zeta^{1/2}]$. Let

$$\theta_t = \arg Z_t \quad \text{and} \quad t_y = \frac{1}{2} \log y. \quad (\text{A.9})$$

For $t \geq t_y$ define $\sigma(t)$ by

$$t = \int_0^{\sigma(t)} \frac{1}{|Z_v|^2} dv + t_y, \quad (\text{A.10})$$

so $d\sigma(t) = |Z_{\sigma(t)}|^2 dt$ and $\sigma(t_y) = 0$. Denote processes under the time change $t = \sigma(t)$ by a star, so $\theta_t^* = \theta_{\sigma(t)}$, etc. By some elementary calculations using Itô's formula (see the proof of [DMS14, Proposition 3.8]), we have $d \log Y_t^* = 2 dt$ and

$$d\theta_t^* = \sqrt{\kappa} \sin \theta_t^* d\widehat{B}_t + \left(2 + \frac{\kappa}{2} - \frac{\rho}{2}\right) \sin(2\theta_t^*) dt \quad (\text{A.11})$$

for \widehat{B}_t a Brownian motion. Since $Y_{t_y}^* = Y_0 = y$, it follows that $Y_t^* = e^{2t}$. Furthermore, as explained in the proof of [DMS14, Proposition 3.8], there is a unique stationary distribution for the SDE (A.11) which takes the form

$$C \sin^\beta(\theta) d\theta, \quad \beta = \frac{8 - 2\rho}{\kappa}, \quad (\text{A.12})$$

where C is a normalizing constant.

Let $\tilde{\theta}_t^*$ be a stationary solution to (A.11). Let $\tilde{Z}_t^* = \frac{e^{2t} e^{i\tilde{\theta}_t^*}}{\sin \theta_t^*}$, so that $\text{Im } \tilde{Z}_t^* = e^{2t}$ and $\arg \tilde{Z}_t^* = \tilde{\theta}_t^*$. Let \widetilde{W}_t^* be determined by \tilde{Z}_t^* in the same manner that W_t^* is determined by Z_t^* . Let

$$\tilde{\sigma}(t) := \int_0^t |\tilde{Z}_v^*|^2 dv.$$

Denote processes under the time change $t = \tilde{\sigma}^{-1}(t)$ by removing the star. Then we have that $(\tilde{\theta}_t, \tilde{Z}_t, \widetilde{W}_t)$ are related in the same manner as (θ_t, Z_t, W_t) . Moreover,

$$\tilde{\sigma}(t) = \inf\{t \in \mathbf{R} : \text{Im } \tilde{Z}_t = e^{2t}\}.$$

Following [DMS14, Section 3], we define a reverse $\text{SLE}_\kappa(\rho)$ process with a force point infinitesimally above 0 to be the Loewner evolution driven by \widetilde{W} .

We will eventually compare reverse $\text{SLE}_\kappa(\rho)$ with force point starting from $(0, y)$ and reverse $\text{SLE}_\kappa(\rho)$ with a force point infinitesimally above 0 by using convergence of a given solution of (A.11) to the stationary distribution. Before we do so, we prove an estimate which is needed to show that the hulls of the reverse $\text{SLE}_\kappa(\rho)$ with force point starting from $(0, y)$ do not get too big during the interval of times before a given solution mixes with the stationary solution.

Lemma A.2. *Let t_y be as in (A.9). For any $p \in (0, 1)$ and $v > 0$, there is a $b > 0$ depending on v, p , and ζ but not ϵ or the particular choice of $y \in [\epsilon, 5\zeta^{1/2}]$ such that*

$$\mathbf{P}_*^z \left(K_{t_y+v}^* \subset B_b(0) \right) \geq 1 - p.$$

Here $K_t^* = K_{\sigma(t)}$, for (K_t) the hulls of the reverse Loewner evolution driven by (W_t) .

Proof. First note that θ_t^* a.s. never hits 0 or π . To see this, one observes that θ_t^* is a time change of a constant multiple of the process of [Law05, Section 1.11] with $a = (4 + \kappa - \rho)/\kappa > 1/2$, so the claim follows from [Law05, Lemma 1.27].

Therefore there exists $\delta > 0$ depending only on v such that if θ_t^* is started at time t_y with initial condition $\theta_{t_y}^* = \pi/2$ then with probability at least $1 - p/2$ we have $\theta_t^* \in (\delta, 2\pi - \delta)$ for each $t \in [t_y, t_y + v]$. Let G be the event that this occurs.

We can find a constant $c > 0$ depending only on δ such that on the event G , we have $X_t^*/Y_t^* \leq c$ for $t \in [t_y, t_y + v]$. It then follows from (A.2) that on this event we have

$$\partial_t Y_t \geq \frac{1}{cY_t}, \quad \forall t \leq \sigma(t_y + v)$$

for a possibly larger c . This implies

$$Y_t^2 \geq c^{-1}t + y^2 \tag{A.13}$$

for a possibly larger constant c . In particular, $(e^{4v} - 1)y^2 = Y_{\sigma(t_y + v)}^2 - y^2 \geq c^{-1}\sigma(t_y + v)$, so for some possibly larger constant c we have

$$\sigma(t_y + v) \leq cy^2. \tag{A.14}$$

Let B_t^z be the Brownian motion of (3.7). We can find a $C > 0$ depending only on ζ such that with probability at least $1 - p/2$, we have $|\sqrt{\kappa}B_t^z| \leq Cy$ for each $t \in [0, cy^2]$. Let G' be the event that this occurs and that G occurs, so that $\mathbf{P}(G') \geq 1 - p$. It follows from (3.7) and (A.14) that on G' , we have

$$\sup_{t \in [0, \sigma(t_y + v)]} |W_t| \preceq 1,$$

with the implicit constant depending only on C . By [Law05, Lemma 4.13] we then have $\text{diam } K_{\sigma(t_y + v)} \preceq 1$. \square

Our next lemma controls the behavior of the Loewner transition maps $\tilde{g}_{\bar{t}, t}^*$ corresponding to a stationary solution to (A.11) after it has been run for a certain amount of time. This estimate will eventually imply an estimate for the analogous transition maps for the Loewner evolution driven by (W_t) by convergence solutions of SDE's to their stationary distribution.

Lemma A.3. *Let (\tilde{g}_t) be the reverse Loewner maps of a reverse $\text{SLE}_\kappa(\rho)$ process with a force point infinitesimally above 0, with hulls (\tilde{K}_t) . We adopt the notation given just above Lemma A.2 so in particular a star denotes processes under the time change $t \mapsto \tilde{\sigma}(t)$. For $\bar{t} \in \mathbf{R}$ and $t \geq \bar{t}$, let $\tilde{g}_{\bar{t}, t}^*$ be the map defined on \mathbf{H} which satisfies $\tilde{g}_t^* = \tilde{g}_{\bar{t}, t}^* \circ \tilde{g}_{\bar{t}}^*$ and let $\tilde{K}_{\bar{t}, t}^* := \tilde{K}_t^* \setminus \tilde{g}_{\bar{t}, t}^*(\tilde{K}_{\bar{t}}^*)$ be the corresponding hull. For $a, d > 0$ and $\mu \in \mathcal{M}$, let $F_{\bar{t}, t} = F_{\bar{t}, t}(a, d, \mu)$ be the event that $\tilde{\sigma}(t) \leq a$ and for each $\delta > 0$, the harmonic measure from ∞ of each of $[-\delta, 0]$ and of $[0, \delta]$ in $\mathbf{H} \setminus (\tilde{K}_{\bar{t}, t}^* \cup \tilde{g}_{\bar{t}, t}^*(B_d(0) \cap \mathbf{H}))$ is at least $\mu(\delta)$. For each $\bar{t}_0 \in \mathbf{R}$, $d > 0$, and $p \in (0, 1)$, we can find $t_* = t_*(\bar{t}_0, d, p) \geq \bar{t}_0$ such that whenever $\bar{t} \leq \bar{t}_0$ and $t \geq t_*$, there exists $a = a(d, p, \bar{t}, \bar{t}_0) > 0$ and $\mu = \mu(d, p, \bar{t}, \bar{t}_0) \in \mathcal{M}$ such that*

$$\mathbf{P}(F_{\bar{t}, t}) \geq 1 - p.$$

Proof. By [DMS14, Proposition 3.10], for each $t > 0$, the conditional law of \tilde{K}_t^* given \tilde{Z}_t^* is that of a forward chordal $\text{SLE}_\kappa(\rho - 8)$ hull with an interior force point at \tilde{Z}_t^* stopped at the first time it hits its force point. By [SW05, Theorem 3] this law is that same as that of the hull of a radial $\text{SLE}_\kappa(\kappa + 2 - \rho)$ from 0 to \tilde{Z}_t^* with a force point at ∞ , run until the first time it hits \tilde{Z}_t^* . Since $\kappa + 2 - \rho > \kappa/2 - 2$ (by our choice of ρ) [MS13, Theorem 1.12] implies that such a process is transient (i.e., almost surely tends to its target point) and [MS13, Lemma 2.4] implies that it a.s. does not intersect itself or hit $\mathbf{R} \cup \{\infty\}$. In particular, \tilde{K}_t^* is a.s. a simple curve which does not intersect \mathbf{R} except at its starting point and has finite half-plane capacity. Therefore the same is a.s. true of $\tilde{K}_{\bar{t}, t}^*$ for each $\bar{t} \in \mathbf{R}$ and $t \geq \bar{t}$.

By uniqueness of the stationary solution to (A.11), for each $v \in \mathbf{R}$ we have $\tilde{\theta}_{\cdot + v}^* \stackrel{d}{=} \tilde{\theta}^*$. Since $\tilde{\theta}^*$ determines the driving function \tilde{W}^* and hence also the Loewner chain (\tilde{g}_t^*) , and since $\tilde{Y}_t^* = e^{2t}$, we have

$$\{e^{-2v}\tilde{g}_{t+v}^*(e^{2v}) : t \in \mathbf{R}\} \stackrel{d}{=} \{\tilde{g}_t^* : t \in \mathbf{R}\}, \quad \forall v \in \mathbf{R}. \tag{A.15}$$

Now fix $\bar{t}_0 \in \mathbf{R}$, $d > 0$, and $p \in (0, 1)$. By (A.15), the law of the diameter of $\tilde{K}_{\bar{t}}^*$ is stochastically non-decreasing as \bar{t} increases. By [Law05, Proposition 3.46], it follows that we can find a deterministic $D = D(\bar{t}_0, d, p) > 0$ such that

$$\mathbf{P}\left((B_d(0) \cap \mathbf{H}) \setminus \tilde{K}_{\bar{t}}^* \subset \tilde{g}_{\bar{t}}^*(B_D(0) \cap \mathbf{H})\right) \geq 1 - p/4 \quad \forall \bar{t} \leq \bar{t}_0. \tag{A.16}$$

Almost surely, the curve $\tilde{K}_{\bar{t}}^*$ does not intersect \mathbf{R} except at its starting point, so there exists some deterministic $\delta > 0$ and $\lambda > 0$ (depending only on \bar{t} and p) such that with probability at least $1 - p/4$, we have $\text{Im } \tilde{g}_0^*(z) \geq \lambda$ for each $z \in B_\delta(0)$. By (A.15), we can find $\mathbf{t}_* = \mathbf{t}_*(\bar{t}_0, D, p, \lambda, \delta) \geq \bar{t}_0$ such that for $\mathbf{t} \geq \mathbf{t}_*$, it holds with probability at least $1 - p/4$ that $\text{Im } \tilde{g}_{\mathbf{t}}^*(w) \geq 1$ for each $w \in B_D(0) \cap \mathbf{H}$.

Suppose $\bar{t} \leq \bar{t}_0$ and $\mathbf{t} \geq \mathbf{t}_*$. If $\text{Im } \tilde{g}_{\mathbf{t}, \mathbf{t}}^*(x) < 1$ for some $x \in B_d(0) \cap \mathbf{H}$, then since $K_{\bar{t}}^*$ has empty interior, there must be some $x' \in (B_d(0) \cap \mathbf{H}) \setminus \tilde{K}_{\bar{t}}^*$ for which $\text{Im } \tilde{g}_{\mathbf{t}, \mathbf{t}}^*(x') < 1$. If the event in (A.16) holds, then $x' = \tilde{g}_{\mathbf{t}}^*(w)$ for some $w \in B_D(0) \cap \mathbf{H}$, so by definition of $\tilde{g}_{\mathbf{t}, \mathbf{t}}^*$ we have $\text{Im } \tilde{g}_{\mathbf{t}}^*(w) < 1$. By our choice of \mathbf{t}_* , we find that

$$\mathbf{P} \left(\text{Im } \tilde{g}_{\mathbf{t}, \mathbf{t}}^*(w) \geq 1, \forall w \in B_d(0) \cap \mathbf{H} \right) \geq 1 - p/2.$$

Since $\tilde{K}_{\mathbf{t}, \mathbf{t}}^* \subset K_{\mathbf{t}}^*$ and $K_{\mathbf{t}}^*$ a.s. does not intersect \mathbf{R} except at 0 and a.s. has finite half plane capacity, for each such $\mathbf{t} \geq \mathbf{t}_*$ we can find a and μ as in the statement of the lemma such $\mathbf{P}(F_{\bar{t}, \mathbf{t}}) \geq 1 - p$ for each $\bar{t} \leq \bar{t}_0$. \square

The following lemma together with Lemma A.1 are the main inputs in the proof of Proposition 3.10.

Lemma A.4. *Suppose we are in the setting of this subsection (so that in particular $X_0 = 0$ and $Y_0 = y$). Let $\tilde{T}_r := \inf\{t \geq 0 : Y_t = r\} = \sigma(\frac{1}{2} \log r)$. Also let $d > 0$ and $p \in (0, 1)$. There is an $r_* > 0$ (depending on ζ, d , and p) such that for $r \geq r_*$, there exists $A > 0$ and $\mu \in \mathcal{M}$, independent of ϵ and the particular choice of $y \in [\epsilon, 5\zeta^{1/2}]$ such that the following is true. Let $E_1 = E_1(r, d, A, \mu)$ be the event that $\tilde{T}_r \leq A$ and for each $\delta > 0$, the harmonic measure from ∞ of each of $[-\delta, 0]$ and of $[0, \delta]$ in $\mathbf{H} \setminus \left(K_{\tilde{T}_r} \cup g_{\tilde{T}_r}(B_d(0) \cap \mathbf{H}) \right)$ is at least $\mu(\delta)$. Then $\mathbf{P}(E_1) \geq 1 - p$.*

Remark A.5. The purpose of the harmonic measure condition in the definition of E_1 is as follows. When we compose with g_{S_0} on the event E_0 of Lemma A.1, the part of the hull grown before time S_0 is “pushed” into $g_{\tilde{T}_r}(B_d(0))$. The harmonic measure condition in the definition of E_1 together with Lemma 2.4 will then imply the occurrence of $G(g_{\tilde{T}_r}^{-1}, \mu)$ on the event $E_0 \cap E_1$. See also Figure A.1.

Proof of Lemma A.4. Define the processes $X_{\mathbf{t}}^*, Y_{\mathbf{t}}^*, Z_{\mathbf{t}}^*, \sigma(\mathbf{t})$, and $\theta_{\mathbf{t}}^*$ as above. Let $(\tilde{g}_{\mathbf{t}})$ be the reverse Loewner maps of a reverse $\text{SLE}_\kappa(\rho)$ process with a force point immediately above 0. We adopt the notation given just above Lemma A.3, so that for $\mathbf{t} > 0$, $\tilde{Z}_{\mathbf{t}}$ is the image of the force point under $\tilde{g}_{\mathbf{t}}$ and $\tilde{\theta}_{\mathbf{t}}^* = \arg \tilde{Z}_{\mathbf{t}}^*$ is the corresponding stationary solution to (A.11).

By convergence of the law of the solution of (A.11) to its stationary distribution, there exists $v > 0$, independent of ϵ and the particular choice of $y \in [\epsilon, 5\zeta^{1/2}]$, such that the following is true. The total variation distance between the law of $\theta_{\mathbf{t}_y+v}^*$, started from $\pi/2$ at time \mathbf{t}_y and the stationary distribution (A.12) is at most $p/4$. Let $\bar{\mathbf{t}}_y = \mathbf{t}_y + v$. We can couple θ^* with $\tilde{\theta}^*$ in such a way that with probability at least $1 - p/3$, these two processes agree at time $\bar{\mathbf{t}}_y$ and (by the Markov property) at every time thereafter. Let F_1 be the event that $\theta_{\mathbf{t}}^* = \tilde{\theta}_{\mathbf{t}}^*$ for each $\mathbf{t} \geq \bar{\mathbf{t}}_y$.

Define the maps $\tilde{g}_{\mathbf{t}_y, \mathbf{t}}^*$ and the hulls $\tilde{K}_{\mathbf{t}_y, \mathbf{t}}^*$ for $\mathbf{t} \geq \bar{\mathbf{t}}_y$ as in Lemma A.3. Define $g_{\mathbf{t}_y, \mathbf{t}}^*$ and $K_{\mathbf{t}_y, \mathbf{t}}^*$ for $\mathbf{t} \geq \bar{\mathbf{t}}_y$ analogously but with $g_{\mathbf{t}}^*$ and $K_{\mathbf{t}}^*$ in place of $\tilde{g}_{\mathbf{t}}^*$ and $\tilde{K}_{\mathbf{t}}^*$. We have that $(\theta_{\mathbf{t}}^*, e^{2\mathbf{t}})$ determines $W_{\mathbf{t}}^*$ and hence also $(g_{\mathbf{t}}^*)$. Similarly for the corresponding processes under the stationary distribution. Therefore on F_1 , we have

$$g_{\mathbf{t}_y, \mathbf{t}}^* = \tilde{g}_{\mathbf{t}_y, \mathbf{t}}^*, \quad K_{\mathbf{t}_y, \mathbf{t}}^* = \tilde{K}_{\mathbf{t}_y, \mathbf{t}}^*, \quad \forall \mathbf{t} \geq \bar{\mathbf{t}}_y. \quad (\text{A.17})$$

By Lemma A.2 we can find a $b > 0$ depending only on v such that the probability of the event

$$F_2 := \{K_{\bar{\mathbf{t}}_y}^* \subset B_b(0)\}$$

is at least $1 - p/3$. By combining this with [Law05, Proposition 3.46], we find that there exists a deterministic constant $d' = d'(d, b) > 0$ such that on the event F_2 we have

$$K_{\mathbf{t}_y}^* \cup g_{\mathbf{t}_y}^*(B_d(0) \cap \mathbf{H}) \subset B_{d'}(0) \cap \mathbf{H}. \quad (\text{A.18})$$

Let $\bar{\mathbf{t}}_0 = 5\zeta^{1/2} + v$, so that $\bar{\mathbf{t}}_y \leq \bar{\mathbf{t}}_0$. Let \mathbf{t}_* be chosen so that the conclusion of Lemma A.3 holds with this choice of $\bar{\mathbf{t}}_0$, d' in place of d , and $p/3$ in place of p . Let $\mathbf{t} \geq \mathbf{t}_*$ and let $a = a(d', p, \mathbf{t}, \bar{\mathbf{t}}_0) > 0$ and

$\mu_0 = \mu_0(d', p, \mathfrak{t}, \bar{\mathfrak{t}}_0) \in \mathcal{M}$ be chosen so that with $F_3 = F_{\bar{\mathfrak{t}}_y, \mathfrak{t}}(a, d', \mu_0)$ the event of Lemma A.3 we have $\mathbf{P}(F_3) \geq 1 - p/3$ for each choice of $\bar{\mathfrak{t}}_y \leq \bar{\mathfrak{t}}_0$. Note that a and μ_0 do not depend on ϵ or the particular choice of $y \in [\epsilon, 5\zeta^{1/2}]$. Then we have

$$\mathbf{P}(F_1 \cap F_2 \cap F_3) \geq 1 - p.$$

If we set $r_* = e^{2\mathfrak{t}_*}$ and $r = e^{2\mathfrak{t}}$, then r ranges over $[r_*, \infty)$ as \mathfrak{t} ranges over $[0, \infty)$. We will now conclude the proof by showing that $F_1 \cap F_2 \cap F_3 \subset E_1$ for an appropriate choice of parameters. On the event $F_1 \cap F_2 \cap F_3$, we have

$$\tilde{T}_r = \text{hcap } K_{\mathfrak{t}}^* \leq \text{hcap } K_{\mathfrak{t}, \bar{\mathfrak{t}}_y}^* + \text{hcap } K_{\bar{\mathfrak{t}}_y}^*.$$

The first term is at most a by the definition of F_3 together with (A.17). The second term is at most a finite constant depending only on b . Hence for $r \geq r_*$ we can find $A > 0$ as in the statement of the lemma such that on $F_1 \cap F_2 \cap F_3$ we have $\tilde{T}_r \leq A$. Furthermore, on $F_1 \cap F_2 \cap F_3$,

$$\begin{aligned} K_{\tilde{T}_r} \cup g_{\tilde{T}_r}^*(B_d(0) \cap \mathbf{H}) &= K_{\mathfrak{t}}^* \cup g_{\mathfrak{t}}^*(B_d(0) \cap \mathbf{H}) \\ &= K_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^* \cup g_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^* \left(K_{\bar{\mathfrak{t}}_y}^* \cup g_{\bar{\mathfrak{t}}_y}^*(B_d(0) \cap \mathbf{H}) \right) \quad (\text{by definition of } g_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^*) \\ &= \tilde{K}_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^* \cup \tilde{g}_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^* \left(K_{\bar{\mathfrak{t}}_y}^* \cup g_{\bar{\mathfrak{t}}_y}^*(B_d(0) \cap \mathbf{H}) \right) \quad (\text{by (A.17)}) \\ &\subset \tilde{K}_{\mathfrak{t}}^* \cup \tilde{g}_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^*(B_{d'}(0) \cap \mathbf{H}) \quad (\text{by (A.18) and the definition of } K_{\bar{\mathfrak{t}}_y, \mathfrak{t}}^*). \end{aligned}$$

It now follows from the definition of F_3 (see Lemma A.3) that for each $r \geq r_*$, we can find $\mu \in \mathcal{M}$ satisfying the conditions of the lemma such that with this choice of μ and A as above, the event E_1 holds on $F_1 \cap F_2 \cap F_3$. \square

A.3 Conclusion of the proof

Now we can combine the results of the previous two subsections to complete the proof of Proposition 3.10.

Proof of Proposition 3.10. Let $\zeta > 0$, $d > 0$, and $p_0 > 0$ be as in Lemma A.1, and let $E_0 = E_0(\zeta, d)$ be the event of that lemma, so that $\mathbf{P}(E_0) \geq p_0$. Let S_0 be as in (A.3) and for $t \geq S_0$, let $g_{S_0, t}$ be the map defined on \mathbf{H} which satisfies $g_t = g_{S_0, t} \circ g_{S_0}$.

Conditional on $\{g_t : t \leq S_0\}$, the law of $\{g_{S_0, v+S_0} : v \geq 0\}$ is the same as that of $\{g_v : v \geq 0\}$ started from $Z_0 = (0, Y_{S_0})$ instead of from $Z_0 = z$. Note that $Y_{S_0} \in [\epsilon, 5\zeta^{1/2}]$ on E_0 . Define the time \tilde{T}_r and the events $E_1 = E_1(r, A, d, \mu)$ as in Lemma A.4 but with $g_{S_0, +S_0}$ in place of g . Let r_* , μ , and A satisfy the conclusion of Lemma A.4 for d as above and $p = 1/2$. Then if $r \geq r_*$ we have $\mathbf{P}(E_1|E_0) \geq 1/2$, whence $\mathbf{P}(E_0 \cap E_1) \geq p_0/2$.

By condition 1 in the definition of E_0 and the definition of E_1 we have $T_r^z = S_0 + \tilde{T}_r \leq \zeta + A$ on $E_0 \cap E_1$. Furthermore, by definition of E_1 , on the event $E_0 \cap E_1$, the harmonic measure from ∞ of each of $[-\delta, 0]$ and $[0, \delta]$ in $\mathbf{H} \setminus K_{T_r^z}$ is at least $\mu(\delta)$. By Lemma 2.4 we can find $\mu' \in \mathcal{M}$ and $t_* > 0$ as in the proposition such that

$$E_0 \cap E_1 \subset \{T_r^z < t_*\} \cap G(g_{T_r^z}^{-1}, \mu').$$

This proves the statement of the proposition. \square

B Comparisons of derivatives using harmonic measure

In this section we will prove some technical lemmas which allow us to compare conformal maps defined on different domains. The results of this section are needed primarily for the proof of the two-point estimate in Section 6. We start with a simple geometric description of the derivative of a certain conformal map defined on a subdomain of \mathbf{D} .

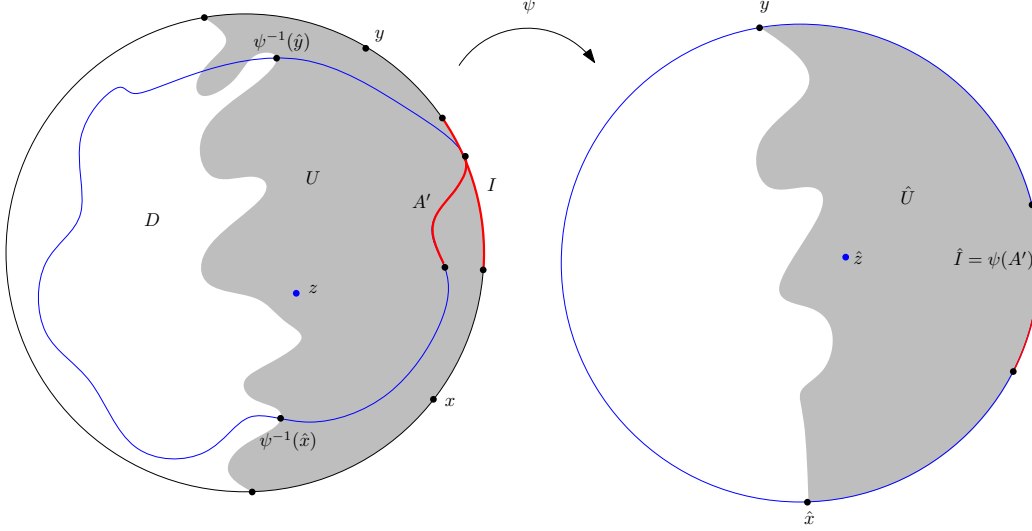


Figure B.1: An illustration of the setup of Lemma B.3. The boundary of D is shown in blue and the arcs I and \hat{I} are shown in red.

Lemma B.1. *Let $U \subset \mathbf{D}$ be a simply connected subdomain with $[x, y]_{\partial \mathbf{D}} \subset \partial U$ and $m \in (x, y)_{\partial \mathbf{D}}$. Let $\Psi : U \rightarrow \mathbf{D}$ be the conformal map taking x to $-i$, y to i , and m to 1 . Let $z \in U$, let I be a sub-arc of $[x, y]_{\partial \mathbf{D}}$, and suppose that for some $\delta > 0$, the distance from $\Psi(z)$ to $\Psi(I)$ and the length of $\Psi(I)$ are each at least δ . Then*

$$\text{hm}^z(I; U) \asymp \text{dist}(z, \partial U) |\Psi'(z)|$$

with the implicit constants depending only δ .

Proof. By conformal invariance of harmonic measure, we have $\text{hm}^z(I; U) = \text{hm}^{\Psi(z)}(\Psi(I); U)$. By our hypotheses on $\Psi(I)$ we have $\text{hm}^{\Psi(z)}(\Psi(I); U) \asymp \text{dist}(\Psi(z), \partial \mathbf{D})$, with the implicit constant depending only on δ . By the Koebe quarter theorem, we have $\text{dist}(\Psi(z), \partial \mathbf{D}) \asymp \text{dist}(z, \partial U) |\Psi'(z)|$ with a universal implicit constant. \square

Remark B.2. We note some circumstances under which the hypotheses of Lemma B.1 are satisfied. Let \hat{U} denote the Schwarz reflection of U across $[x, y]_{\partial \mathbf{D}}$. Suppose $I \subset (x, y)_{\partial \mathbf{D}}$ with $m \in I$ and the distance from $\partial U \setminus \partial \mathbf{D}$ to I is at least a constant $\zeta > 0$. If z lies at distance at least a constant $\zeta' > 0$ from $\partial \mathbf{D}$ and is sufficiently close to ∂U , then by considering harmonic measure from m in \hat{U} (c.f. the proof of Lemma 2.8), we get that the hypotheses of Lemma B.1 are satisfied with δ depending only on ζ, ζ' and the length of I . In particular, if the event $\mathcal{G}_{[x, y]_{\partial \mathbf{D}}}(\Psi, \mu)$ of Section 2.2.2 occurs, then Lemma 2.8 implies that, under the same hypotheses on z , the hypotheses of Lemma B.1 are satisfied with δ depending only on μ, ζ' , and the length of I .

From Lemma B.1, we can deduce some additional lemmas which allow us to compare the derivatives of conformal maps on different domains.

Lemma B.3. *Let $U \subset \mathbf{D}$ be a simply connected subdomain with $[x, y]_{\partial \mathbf{D}} \subset \partial U$. Fix constants $\zeta, \delta, \Delta > 0$. Let $m \in (x, y)_{\partial \mathbf{D}}$ with $|x - m|, |y - m| \geq \Delta > 0$. Let $\phi : U \rightarrow \mathbf{D}$ be the conformal map taking x to $-i$, y to i , and m to 1 .*

Let $z \in U$ and $\hat{z} \in \mathbf{D}$ with $1 - |\hat{z}| \geq \zeta$. Let $D \subset \mathbf{D}$ be another subdomain containing z with $\text{dist}(z, \partial D) \geq \zeta$. Let $\psi : D \rightarrow \mathbf{D}$ be a conformal map which takes z to \hat{z} . Let \hat{U} be the connected component of $\psi(U \cap D)$ containing \hat{z} . Suppose there is a connected arc A of ∂D which disconnects z from $[x, y]_{\partial \mathbf{D}}$ in U . Let $[\hat{x}, \hat{y}]_{\partial \mathbf{D}} = \psi(A)$. Suppose there exists $\hat{m} \in (\hat{x}, \hat{y})_{\partial \mathbf{D}}$ with $|\hat{x} - \hat{m}|, |\hat{y} - \hat{m}| \geq \Delta$. Let $\hat{\phi} : \hat{U} \rightarrow \mathbf{D}$ be the conformal map taking \hat{x} to $-i$, \hat{y} to i , and \hat{m} to 1 .

Let A' be an arc of ∂U contained in A and let I be an arc of $[x, y]_{\partial \mathbf{D}}$. Suppose $\delta > 0$ and $\mu \in \mathcal{M}$ are such that the following hold.

1. The length of $\phi(I)$ and the distance from $\phi(z)$ to $\phi(I)$ are each at least δ .
2. The distance from $\widehat{\phi}(\widehat{z})$ to $[-i, i]_{\partial \mathbf{D}}$ is at least δ .
3. $\text{hm}^z(A'; U) \geq \delta$.
4. The probability that a Brownian motion started from any point of A' exits U in I is at least δ .
5. $\mathcal{G}_{[\widehat{x}, \widehat{y}]_{\partial \mathbf{D}}}(\widehat{\phi}, \mu)$ occurs (Definition 2.3).

Then we have

$$|\phi'(z)| \asymp |\widehat{\phi}'(\widehat{z})|, \quad (\text{B.1})$$

with implicit constants depending only on $\zeta, \delta, \mu, \Delta, z$, and \widehat{z} and uniform for z and \widehat{z} in compact subsets of \mathbf{D} .

See Figure B.1 for an illustration of the setup.

Proof. Throughout, we assume that all implicit constants depend only on ζ, δ, Δ, z , and \widehat{z} and are uniform for z and \widehat{z} in compact subsets of \mathbf{D} .

By Lemma B.1 and assumptions 1 and 2, we have

$$|\phi'(z)| \asymp \frac{\text{hm}^z(I; U)}{\text{dist}(z; \partial U)}, \quad |\widehat{\phi}'(\widehat{z})| \asymp \frac{\text{hm}^{\widehat{z}}([\widehat{x}, \widehat{y}]_{\partial \mathbf{D}}; \widehat{U})}{\text{dist}(\widehat{z}; \partial \widehat{U})}. \quad (\text{B.2})$$

By the Koebe quarter theorem,

$$1 \asymp \frac{\text{dist}(\widehat{z}, \partial \mathbf{D})}{\text{dist}(z, \partial U)} \asymp |\psi'(z)| \asymp \frac{\text{dist}(\widehat{z}, \partial \widehat{U})}{\text{dist}(z, \partial U)}.$$

Thus

$$\text{dist}(z, \partial U) \asymp \text{dist}(\widehat{z}, \partial \widehat{U}). \quad (\text{B.3})$$

By conformal invariance of harmonic measure, we have

$$\text{hm}^{\widehat{z}}([\widehat{x}, \widehat{y}]_{\partial \mathbf{D}}; \widehat{U}) = \text{hm}^z(A; \psi^{-1}(\widehat{U})). \quad (\text{B.4})$$

By our assumption on A , a Brownian motion started from z must exit $\psi^{-1}(\widehat{U})$ in A before leaving ∂U in $[x, y]_{\partial \mathbf{D}}$. Hence

$$\text{hm}^z(I, U) \leq \text{hm}^z(A; \psi^{-1}(\widehat{U})).$$

By combining (B.2), (B.3), and (B.4) we get $|\phi'(z)| \leq |\widehat{\phi}'(\widehat{z})|$.

For the reverse inequality, let $\widehat{I} = \psi(A')$. By assumptions 3 and 5, the length of $\widehat{\phi}(\widehat{I})$ is ≥ 1 . By assumption 2 and Lemma B.1 we then get

$$|\widehat{\phi}'(\widehat{z})| \asymp \frac{\text{hm}^z(\widehat{I}; \widehat{U})}{\text{dist}(\widehat{z}; \partial \widehat{U})}.$$

By conformal invariance of harmonic measure and (B.3) this is proportional to

$$\frac{\text{hm}^z(A'; \psi^{-1}(\widehat{U}))}{\text{dist}(z, U)}.$$

By assumption 4 and the first proportionality in (B.2) we get that this last quantity is $\asymp |\phi'(z)|$. \square

Lemma B.4. *Let $x, y \in \partial \mathbf{D}$. Let $\eta : [0, \infty] \rightarrow \mathbf{D}$ be a simple curve which does not intersect $(x, y)_{\partial \mathbf{D}}$. Let $m \in (x, y)_{\partial \mathbf{D}}$ with $|x - m|, |y - m| \geq \Delta > 0$. Let D_η be the connected component of $\mathbf{D} \setminus \eta$ containing $[x, y]_{\partial \mathbf{D}}$ on its boundary. Let $\Psi_\eta : D_\eta \rightarrow \mathbf{D}$ be the conformal map taking x to $-i$, y to i , and m to 1 . Let $t_2 > t_1 \geq 0$. Let $D_\eta^0 = \mathbf{D} \setminus (\eta([0, t_1]) \cup \eta([t_2, \infty)))$. Let $\Phi : D_\eta^0 \rightarrow \mathbf{D}$ be the conformal map taking x^+ to $-i$, y^- to i , and m to 1 . Let $I \subset [x, y]_{\partial \mathbf{D}}$ be an arc. Let $z \in D_\eta$. Suppose that there is some $\ell > 0$ and $\delta > 0$ such that the following is true.*

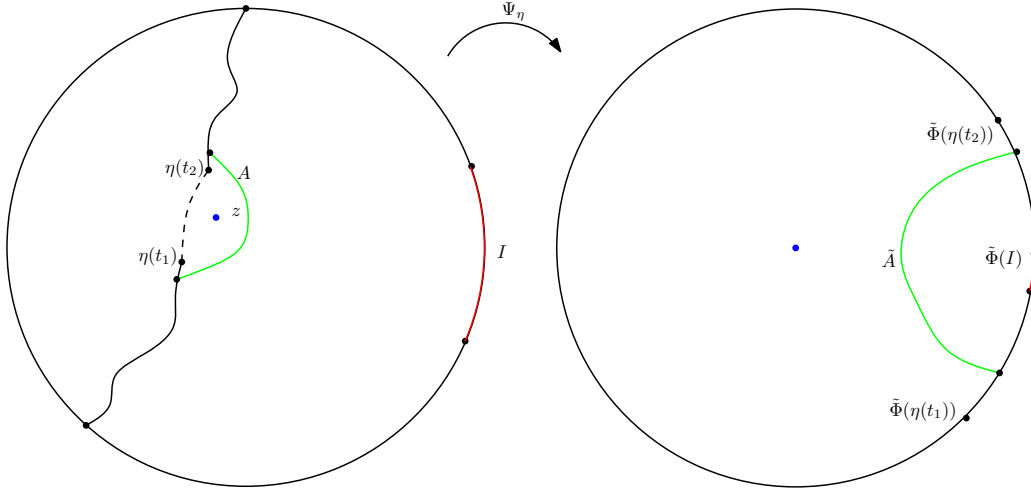


Figure B.2: An illustration of the proof of Lemma B.4. The probability that a Brownian motion started from z exits D_η^0 in the red arc I is bounded by the supremum of the harmonic measure of I in D_η^0 from any point of the green crosscut A . This, in turn, is bounded by a constant times the supremum of the harmonic measure of I in D_η from any point of A , which is bounded by the harmonic measure of I from z in D_η by our choice of \tilde{A} .

1. $\text{hm}^z(\eta([0, t_1]); D_\eta)$ and $\text{hm}^z(\eta([t_2, \infty]); D_\eta)$ are each at least ℓ .
2. The length of $\Psi_\eta(I)$ and the distance from $\Psi_\eta(z)$ to $\Psi_\eta(I)$ are each at least δ .
3. The length of $\Phi(I)$ and the distance from $\Phi(z)$ to $\Phi(I)$ are each at least δ .

Then $|\Phi'(z)| \asymp |\Psi'_\eta(z)|$ and $\text{dist}(z, \partial D_\eta) \asymp \text{dist}(z, \partial D_\eta^0)$ with implicit constants depending only on δ , ℓ , and z , but uniform for z in compact subsets of \mathbf{D} .

Proof. See Figure B.2 for an illustration of the proof.

By Lemma B.1,

$$|\Phi'(z)| \asymp \frac{\text{hm}^z(I; D_\eta^0)}{\text{dist}(z, \partial D_\eta^0)}, \quad |\Psi'_\eta(z)| \asymp \frac{\text{hm}^z(I; D_\eta)}{\text{dist}(z, \partial D_\eta)}$$

with the implicit constants depending only on δ . We clearly have $\text{hm}^z(I; D_\eta^0) \geq \text{hm}^z(I; D_\eta)$. By the Beurling estimate, $\text{hm}^z(\eta \cap B_{r \text{dist}(z, \eta)}(z); D_\eta) \rightarrow 1$ as $r \rightarrow \infty$, at a rate which does not depend on η or $\text{dist}(z, \eta)$. So, our hypothesis 1 implies that $\text{dist}(z, \partial D_\eta) \asymp \text{dist}(z, \partial D_\eta^0)$. Therefore it is enough to prove

$$\text{hm}^z(I; D_\eta^0) \preceq \text{hm}^z(I; D_\eta) \tag{B.5}$$

with the implicit constant depending only on ℓ .

Let $\tilde{\Psi}_\eta : D_\eta \rightarrow \mathbf{D}$ be the conformal map taking z to 0 and m to 1. By conformal invariance of harmonic measure and our hypothesis 1, the distance from each of $\tilde{\Psi}_\eta(\eta(t_1))$ and $\tilde{\Psi}_\eta(\eta(t_2))$ to $\tilde{\Psi}_\eta(I)$ is at least $2\pi\ell$. Hence we can choose a crosscut \tilde{A} in \mathbf{D} which disconnects 0 from $\tilde{\Psi}_\eta(I)$ such that each point of \tilde{A} lies at distance at least ℓ from $\tilde{\Psi}_\eta(I)$ and from $[\tilde{\Psi}_\eta(\eta(t_2)), \tilde{\Psi}_\eta(\eta(t_1))]\partial\mathbf{D}$. The harmonic measure of $\tilde{\Psi}_\eta(I)$ from each point of \tilde{A} in \mathbf{D} is bounded above by a constant depending only on ℓ times the length of $\tilde{\Psi}_\eta(I)$, which in turn is proportional to $\text{hm}^z(I; D_\eta)$. Furthermore, the harmonic measure of the arc $[\tilde{\Psi}_\eta(\eta(t_2)), \tilde{\Psi}_\eta(\eta(t_1))]\partial\mathbf{D}$ from each point of \tilde{A} in \mathbf{D} is bounded above by a constant $a < 1$ depending only on ℓ .

Let $A = \tilde{\Psi}_\eta^{-1}(\tilde{A})$. Then we have

$$\text{hm}^w(I; D_\eta) \preceq \text{hm}^z(I; D_\eta), \quad \text{hm}^w(\eta([t_1, t_2]); D_\eta) \leq a \quad \forall w \in A \tag{B.6}$$

with the implicit constant depending only on ℓ .

A Brownian motion started from z must hit A before exiting D_η^0 in I . Therefore,

$$\text{hm}^z(I; D_\eta^0) \leq \sup_{w \in A} \text{hm}^w(I; D_\eta^0). \quad (\text{B.7})$$

For $w \in A$, we can decompose the event that a Brownian motion B started at w exits D_η^0 in I as the union of the event that B hits I before $\eta([t_1, t_2])$ and the event that B hits $\eta([t_1, t_2])$ and then I . By (B.6) the former event has probability at most a constant C (depending only on ℓ) times $\text{hm}^z(I; D_\eta)$. By the Markov property the latter event has probability at most

$$\sup_{w \in A} \text{hm}^w(\eta([t_1, t_2]); D_\eta) \sup_{v \in \eta([t_1, t_2])} \text{hm}^v(I; D_\eta^0).$$

Since A disconnects $\eta([t_1, t_2])$ from I in D_η^0 we have $\sup_{v \in \eta([t_1, t_2])} \text{hm}^v(I; D_\eta^0) \leq \sup_{w \in A} \text{hm}^w(I; D_\eta^0)$. By combining this with (B.6) we get

$$\sup_{w \in A} \text{hm}^w(I; D_\eta^0) \leq C \text{hm}^z(I; D_\eta) + a \sup_{w \in A} \text{hm}^w(I; D_\eta^0). \quad (\text{B.8})$$

Since $a < 1$, we can re-arrange the estimate (B.8) to get

$$\sup_{w \in A} \text{hm}^w(I; D_\eta^0) \leq \text{hm}^z(I; D_\eta),$$

which together with (B.7) yields (B.5). \square

C Strict mutual absolute continuity for SLE

Definition C.1. We say that a measure μ is *strictly mutually absolutely continuous* (s.m.a.c.) with respect to a measure ν if μ and ν are mutually absolutely continuous with Radon-Nikodym derivative a.e. bounded above and below by finite and positive constants.

In this appendix we will prove a lemma which gives that the conditional law of the “middle part” of an $\text{SLE}_\kappa(\rho^L; \rho^R)$ curve, given a suitable realization of its initial and terminal segments, is s.m.a.c. with respect to the law of the middle part of an ordinary SLE_κ curve (see Lemma C.4 below for an exact statement). This result is needed in the proof of our two-point estimate. We will deduce our desired result from [MW14, Lemma 2.8] (which gives a similar strict mutual absolute continuity statement for $\text{SLE}_\kappa(\rho)$ curves in domains which agree in a neighborhood of the starting point) together with the coupling results of [MS16a], described in Section 2.5.

Before we can prove this result, we need to define what we mean by a “suitable realization of the initial and terminal segments of the path.” Let $x, y \in \partial \mathbf{D}$ be distinct. Let η be a random curve from x to y in \mathbf{D} , with time reversal $\bar{\eta}$. In what follows, we write $\mathcal{B}_\beta = B_{e^{-\beta}}(0)$ and let τ_β (resp. $\bar{\tau}_\beta$) be the first time η (resp. $\bar{\eta}$) hits \mathcal{B}_β , as in Section 6.1.

Fix $\Delta > \Delta' > \tilde{\Delta} > 0$. Suppose we are given times $\sigma, \bar{\sigma} > 0$. Let η^* be the part of η between $\eta(\sigma)$ and $\bar{\eta}(\bar{\sigma})$. Let $H^* = H^*(\eta^*; \tilde{\Delta})$ be the event that $\eta^* \subset \mathcal{B}_{\tilde{\Delta}}$. Let $S = S(\eta; \sigma, \bar{\sigma}, \Delta, \tilde{\Delta})$ be the event that the following occur.

1. $\tau_\Delta \leq \sigma < \infty$ and $\bar{\tau}_\Delta \leq \bar{\sigma} < \infty$ (here, $\tau_\Delta = \tau_\beta$ and $\bar{\tau}_\Delta = \bar{\tau}_\beta$ with $\beta = \Delta$).
2. η^σ (resp. $\bar{\eta}^{\bar{\sigma}}$) is contained in the $e^{-2\Delta}$ -neighborhood of the segment $[x, 0]$ (resp. $[y, 0]$).
3. The conditional probability of H^* given $\eta^\sigma \cup \bar{\eta}^{\bar{\sigma}}$ is positive.

Also let $S^* = S^*(\eta; \sigma, \bar{\sigma}, \Delta, \Delta', \tilde{\Delta})$ be the event that the following occur.

1. $S(\eta; \sigma, \bar{\sigma}, \Delta, \tilde{\Delta})$ occurs.
2. $\eta([\tau_{\Delta'}, \sigma])$ (resp. $\bar{\eta}([\bar{\tau}_{\Delta'}, \bar{\sigma}])$) is contained in $\mathcal{B}_{\tilde{\Delta}}$.

Remark C.2. If the event $L_{z,n}$ and the times $\sigma_{z,n}$ and $\bar{\sigma}_{z,n}$ are defined as in Section 6.2, then we have

$$L_{z,n} \subset S^*(\eta_{z,n}^0; \sigma_{z,n}, \bar{\sigma}_{z,n}, \Delta, \Delta/2, \tilde{\Delta}).$$

This is our primary interest in the event $S^*(\cdot)$.

Remark C.3. In the case that η is an $\text{SLE}_\kappa(\rho^L; \rho^R)$ (which is what we consider in the section) one can show that condition 3 in the definition of S is in fact implied by the other conditions in the definition of S . The idea to establish this is to realize η as a flow line of a GFF, then condition on two counterflow lines (run up to a certain stopping time) with the property that the interface between them is a.s. equal to $\bar{\eta}^\sigma$. See [MS16b, Section 5.4] for a similar argument. We do not need this fact here though, so for the sake of brevity we include condition 3 as a condition.

The main result of this section is the following.

Lemma C.4. Let $\rho^L, \rho^R \in (-2, 0]$, $\delta > 0$, and $x, y \in \partial \mathbf{D}$ with $|x - y| \geq \delta$. Let η be a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ process from x to y in \mathbf{D} with force points located at x^- and x^+ . Let $\bar{\eta}$ be its time reversal. Let σ be a stopping time for η and let $\bar{\sigma}$ be a stopping time for the filtration generated by η^σ and $\bar{\eta}$. Let $S^* = S^*(\eta; \sigma, \bar{\sigma}, \Delta, \Delta', \tilde{\Delta})$ as above. Also let η^* and $H^* = H^*(\eta^*; \tilde{\Delta})$ be as above. Let D be the connected component of $\mathbf{D} \setminus (\eta^\sigma \cup \bar{\eta}^{\bar{\sigma}})$ containing 0.

If $\tilde{\Delta}$ (and hence also Δ' and Δ) is chosen sufficiently large, then the conditional law of η^* given a.e. realization of $\eta^\sigma \cup \bar{\eta}^{\bar{\sigma}}$ for which S^* occurs and the event H^* is s.m.a.c. with respect to the law of a chordal SLE_κ from $\eta(\sigma)$ to $\bar{\eta}(\bar{\sigma})$ in D conditioned on H^* , with deterministic constants depending only on $\rho^L, \rho^R, \kappa, \Delta, \Delta', \tilde{\Delta}$, and δ .

Remark C.5. The statement of Lemma C.4 is also true for general values of $\rho^L, \rho^R \in (-2, \infty)$. We give a full proof here only in the case of negative ρ^L, ρ^R because this is the case which we will need in this paper. We give here a brief outline of the adaptations necessary to treat the case of a general choice of $\rho^L, \rho^R \in (-2, \infty)$. The proof in the case when both ρ^L and ρ^R are positive proceeds in the same manner as the proof of Lemma C.4, except that we condition on auxiliary flow lines to turn a $\text{SLE}_\kappa(\rho^L; \rho^R)$ into an ordinary SLE_κ , instead of turning an ordinary SLE_κ into an $\text{SLE}_\kappa(\rho^L; \rho^R)$ as in the argument below. The case where ρ^L and ρ^R have different signs follows by starting with an $\text{SLE}_\kappa(\rho_0^L; \rho_0^R)$ process with $\rho_0^L, \rho_0^R > 0$, conditioning on auxiliary flow lines to turn it into an $\text{SLE}_\kappa(\rho^L; \rho^R)$, and using the case of positive ρ^L, ρ^R .

For the proof of Lemma C.4, we will assume neither ρ^L nor ρ^R is equal to 0; the case when one of the force points is equal to 0 is treated similarly but with only a single auxiliary flow line.

Choose $\Delta_0 > \tilde{\Delta}_0 > 0$ satisfying $\tilde{\Delta}_0 < \tilde{\Delta} < \Delta' < \Delta_0 < \Delta$, with Δ, Δ' , and $\tilde{\Delta}$ as in the statement of Lemma C.4. Let η_0 be an ordinary chordal SLE_κ from x to y in \mathbf{D} . Let $\bar{\eta}_0$ be the time reversal of η_0 . Let σ_0 (resp. $\bar{\sigma}_0$) be the first time η_0 (resp. $\bar{\eta}_0$) hits $\mathcal{B}_{\tilde{\Delta}}$. Let η_0^* be the part of η_0 between $\eta(\sigma_0)$ and $\bar{\eta}(\bar{\sigma}_0)$. Also let

$$S_0 := S(\eta_0; \sigma_0, \bar{\sigma}_0, \Delta_0, \tilde{\Delta}_0), \quad H_0^* = H^*(\eta_0^*; \tilde{\Delta}_0). \quad (\text{C.1})$$

We can couple η_0 with a GFF h on \mathbf{D} with appropriately chosen boundary data in such a way that η_0 is the zero angle flow line⁸ (in the sense of Section 2.5) of h started from x . Let $\theta^L > 0$ and $\theta^R < 0$ be chosen so that

$$\frac{\theta^L \chi}{\lambda} - 2 = \rho^L, \quad -\frac{\theta^R \chi}{\lambda} - 2 = \rho^R. \quad (\text{C.2})$$

Let η_- and η_+ be the flow lines of h started from x with angles θ^L and θ^R , respectively. Since $\rho^L, \rho^R \in (-2, 0)$, the flow lines η_- and η_+ are well defined. Let D_0 be the connected component of $\mathbf{D} \setminus (\eta_- \cup \eta_+)$ containing the origin. Let b and \bar{b} , respectively, be the first and last point on ∂D_0 hit by η_0 . By the results of [MS16a, Section 7], the conditional law of the part of η_0 which lies in D_0 given $\eta_- \cup \eta_+$ is that of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ process with force points located on either side of b .

Fix $\alpha \in (0, 1)$. Let t_- and t_+ respectively be the first times η_- and η_+ exit $B_{1-\alpha}(0)$.

Throughout the remainder of this subsection, we require all implicit constants, including those in s.m.a.c., to depend only on $\Delta, \tilde{\Delta}, \Delta', \Delta_0, \tilde{\Delta}_0, \alpha, \rho^L, \rho^R, \kappa$, and δ (in particular, implicit constants are not allowed to depend on the realization of whatever we are conditioning on or on the choice of stopping times $\sigma, \bar{\sigma}$).

⁸In the case $\kappa = 4$, we replace flow lines by level lines, as defined in [SS13, SS09]. Everything works the same with this replacement.

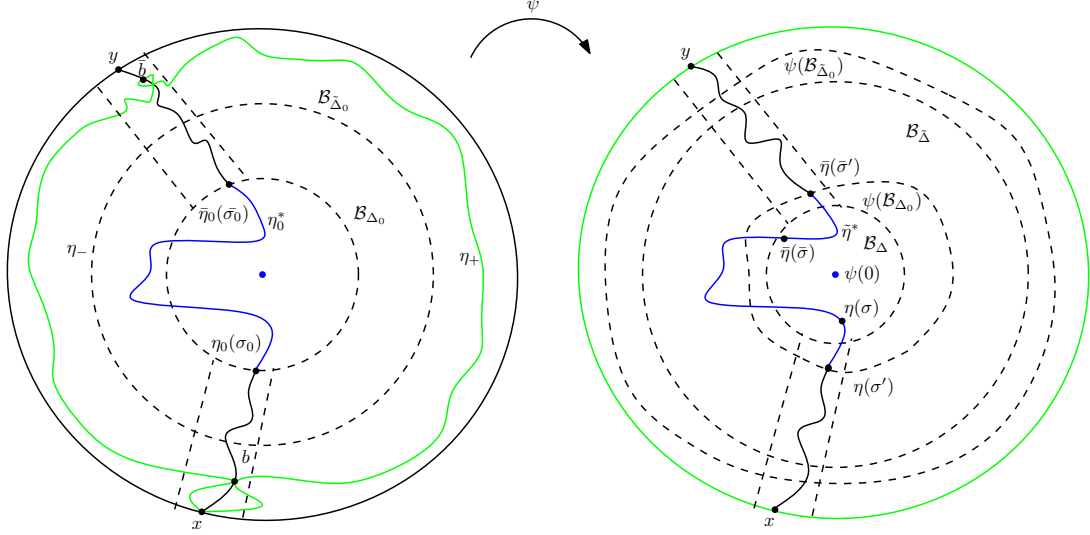


Figure C.1: An illustration of the setup for the proof of Lemma C.4. The curve η_0 in the left picture has the law of an ordinary chordal SLE_κ from x to y in \mathbf{D} . The curve η in the right picture (obtained by mapping the “pocket” D_0 formed green auxiliary flow lines to \mathbf{D}) has the law of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ from x to y . The amount by which ψ distorts distances is exaggerated for clarity— typically, ψ is close to the identity on the event F .

Lemma C.6. *Let ω_0 be a realization of $\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0}$ for which S_0 occurs. If $\tilde{\Delta}_0$ (and hence also Δ_0) is chosen sufficiently large and $\alpha > 0$ is chosen sufficiently small then the following is true for a.e. such ω_0 . For a.e. realization of $(\eta_-^{t-}, \eta_+^{t+})$, the conditional law of η_0^* given ω_0 , H_0^* , and $(\eta_-^{t-}, \eta_+^{t+})$ is s.m.a.c. with respect to the conditional law of η_0^* given only ω_0 and H_0^* .*

Proof. Let \mathbf{P}_{ω_0} denote the regular conditional probability given ω_0 and H_0^* . Let A_0^* be an event with positive \mathbf{P}_{ω_0} -probability which is measurable with respect to η_0 and contained in H_0^* . Let A_0^F be the intersection of H_0^* with an event which is measurable with respect to $\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0}$ and $(\eta_-^{t-}, \eta_+^{t+})$ and contained in S_0 . By Bayes’ rule,

$$\mathbf{P}_{\omega_0}(A_0^* | A_0^F) = \frac{\mathbf{P}_{\omega_0}(A_0^F | A_0^*) \mathbf{P}_{\omega_0}(A_0^*)}{\mathbf{P}_{\omega_0}(A_0^F)}. \quad (\text{C.3})$$

Hence we are lead to study the conditional law of $(\eta_-^{t-}, \eta_+^{t+})$ given ω_0 and η_0^* , for varying realizations of η_0^* for which H_0^* occurs.

By the results of [MS16a, Section 7.1], the conditional law of η_+ given η_0 is that of a chordal $\text{SLE}_\kappa(\rho_F^L; \rho_F^R)$ process from x to y in the right connected component of $\mathbf{D} \setminus \eta_0$ for certain $\rho_F^L, \rho_F^R \in \mathbf{R}$ depending on ρ^L and ρ^R . A similar statement holds for η_- . Furthermore, η_+ and η_- are conditionally independent given η_0 . By [MW14, Lemma 2.8] and condition 2 in the definition of S_0 , if $\tilde{\Delta}_0$ is chosen sufficiently large and $\alpha > 0$ is chosen sufficiently small then the conditional law of the pair $(\eta_-^{t-}, \eta_+^{t+})$ given ω_0 and η_0^* for varying realizations of η_0^* for which H_0^* occurs are s.m.a.c.. By averaging over all such realizations, we get $\mathbf{P}_{\omega_0}(A_0^F | A_0^*) \asymp \mathbf{P}_{\omega_0}(A_0^F)$. By (C.3) we therefore have $\mathbf{P}_{\omega_0}(A_0^* | A_0^F) \asymp \mathbf{P}_{\omega_0}(A_0^*)$. \square

Proof of Lemma C.4. See Figure C.1 for an illustration of the argument.

Let ω_0 be a realization of $\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0}$ for which S_0 occurs and let \mathbf{P}_{ω_0} denote the regular conditional probability given ω_0 and H_0^* , as in Lemma C.6.

Let D_0 , b , and \bar{b} be defined as in the discussion just above Lemma C.6. Let $\psi : D_0 \rightarrow \mathbf{D}$ be the conformal map which takes b to x and \bar{b} to y , chosen so that $|\psi(0)|$ is minimal amongst all such maps. Let

$$\eta := \psi(\eta_0 \cap D_0), \quad \tilde{\eta}^* := \psi(\eta_0^*).$$

By the discussion just above Lemma C.6, the conditional law of η given ω_F is that of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ process from x to y in \mathbf{D} .

Fix $\epsilon > 0$, to be chosen later, and let F be the event that the following occur.

1. η_- and η_+ trace all of ∂D_0 before times t_- and t_+ .
2. $|\psi(z) - z| \leq \epsilon$ for each $z \in D_0$.

By [MW14, Lemma 2.5], we have that $\mathbf{P}_{\omega_0}(F) > 0$ for any choice of $\epsilon > 0$ and a.e. choice of realization ω_0 .

By choosing $\epsilon > 0$ sufficiently small (depending only on $\Delta, \Delta', \tilde{\Delta}, \Delta_0$, and $\tilde{\Delta}_0$), we can arrange that the following are true on F .

1. We have $\mathcal{B}_\Delta \subset \psi(\mathcal{B}_{\Delta_0}) \subset \psi(\mathcal{B}_{\Delta'}) \subset \psi(\mathcal{B}_{\tilde{\Delta}_0}) \subset \mathcal{B}_{\tilde{\Delta}}$.
2. The image under ψ of the $e^{-2\Delta_0}$ -neighborhood of the segment $[x, 0]$ (resp. $[y, 0]$) contains the $e^{-2\Delta}$ -neighborhood of the segment $[x, 0]$ (resp. $[y, 0]$).

On the event F , let σ' and $\bar{\sigma}'$ be the stopping times for η and $\bar{\eta}$ corresponding to σ_0 and $\bar{\sigma}_0$, so $\psi(\eta_0(\sigma_0)) = \eta(\sigma')$, $\psi(\bar{\eta}_0(\bar{\sigma}_0)) = \bar{\eta}(\bar{\sigma}')$, and $\tilde{\eta}^*$ is the part of η between $\eta(\sigma')$ and $\bar{\eta}(\bar{\sigma}')$. Also let η^* be the part of η between σ and $\bar{\sigma}$, as in the statement of the lemma.

By conditions 1 and 2 above together with condition 2 in the definition of S^* , we have

$$F \cap S^* \cap H^* \subset F \cap S_0 \cap H_0^*. \quad (\text{C.4})$$

(Note that the first inclusion is the only place where we use condition 2 in the definition of S^* .) Furthermore, by the first inclusion in condition 1 and condition 1 in the definition of S , on $F \cap S$ we a.s. have

$$\sigma' \leq \tau_\Delta \leq \sigma \quad \text{and} \quad \bar{\sigma}' \leq \bar{\tau}_\Delta \leq \bar{\sigma}. \quad (\text{C.5})$$

Let (ω_0, ω_F) be a realization of $(\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0}, \eta_+^{t_+} \cup \eta_-^{t_-})$ for which $F \cap S_0$ occurs. We observe the following.

1. By the Markov property, the law of η_0^* given ω_0 and H_0^* is that of a chordal SLE_κ from $\eta_0(\sigma_0)$ to $\bar{\eta}_0(\bar{\sigma}_0)$ in $\mathbf{D} \setminus (\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0})$, conditioned on H_0^* .
2. It therefore follows from Lemma C.6 that the law of η_0^* given (ω_0, ω_F) and H_0^* is a.s. s.m.a.c. with respect to the law of a chordal SLE_κ from $\eta_0(\sigma_0)$ to $\bar{\eta}_0(\bar{\sigma}_0)$ in $\mathbf{D} \setminus (\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0})$, conditioned on H_0^* .
3. By [MW14, Lemma 2.8], this latter law is s.m.a.c. with respect to the law of a chordal SLE_κ from $\eta_0(\sigma_0)$ to $\bar{\eta}_0(\bar{\sigma}_0)$ in the connected component of $D_0 \setminus (\eta_0^{\sigma_0} \cup \bar{\eta}_0^{\bar{\sigma}_0})$ containing 0, conditioned on H_0^* .
4. Therefore, the conditional law of $\tilde{\eta}^*$ given (ω_0, ω_F) and H_0^* is s.m.a.c. with respect to the law of a chordal SLE_κ from $\eta(\sigma')$ to $\bar{\eta}(\bar{\sigma}')$ in the component of $\mathbf{D} \setminus (\eta^{\sigma'} \cup \bar{\eta}^{\bar{\sigma}'})$ containing 0, conditioned on H_0^* .
5. By (C.4), (C.5), and the Markov property and reversibility of ordinary SLE_κ , assertion 4 implies that the conditional law of η^* given ω_F ; a realization of $\eta^\sigma \cup \bar{\eta}^{\bar{\sigma}}$ for which S^* occurs; and H^* is a.s. s.m.a.c. with respect to the law of a chordal SLE_κ from $\eta(\sigma')$ to $\bar{\eta}(\bar{\sigma}')$ in the component of $\mathbf{D} \setminus (\eta^{\sigma'} \cup \bar{\eta}^{\bar{\sigma}'})$ containing 0, conditioned on H^* .

Since the law of η given a.e. ω_F is that of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ from x to y in \mathbf{D} and there is a positive probability event of choices for ω_F , assertion 5 implies the statement of the lemma. \square

References

- [ABJ15] T. Alberts, I. Binder, and F. Johansson Viklund. A Dimension Spectrum for SLE Boundary Collisions. *ArXiv e-prints*, January 2015, 1501.06212.
- [BD14] I. Binder and B. Duplantier. Private communication, 2014.

- [Bef08] V. Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008, math/0211322. MR2435854 (2009e:60026)
- [BS05] D. Beliaev and S. Smirnov. Harmonic measure on fractal sets. In *European Congress of Mathematics*, pages 41–59. Eur. Math. Soc., Zürich, 2005. MR2185735 (2007d:31013)
- [BS09] D. Beliaev and S. Smirnov. Harmonic measure and SLE. *Comm. Math. Phys.*, 290(2):577–595, 2009, 0801.1792. MR2525631 (2011c:60265)
- [DB02] B. Duplantier and I. Binder. Harmonic measure and winding of conformally invariant curves. *Physical Review Letters*, 89(264101), 2002, cond-mat/0208045.
- [DB08] B. Duplantier and I. Binder. Harmonic measure and winding of random conformal paths: A Coulomb gas perspective. *Nucl. Phys. B*, 802:494–513, 2008, 0802.2280.
- [DHBZ15] B. Duplantier, X. Hieu Ho, T. Binh Le, and M. Zinsmeister. Logarithmic Coefficients and Multifractality of Whole-Plane SLE. *ArXiv e-prints*, April 2015, 1504.05570.
- [DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *ArXiv e-prints*, September 2014, 1409.7055.
- [DNNZ12] B. Duplantier, C. Nguyen, N. Nguyen, and M. Zinsmeister. The Coefficient Problem and Multifractality of Whole-Plane SLE and LLE. *ArXiv e-prints*, November 2012, 1211.2451.
- [DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011, 1206.0212. MR2819163 (2012f:81251)
- [Dub09a] J. Dubédat. Duality of Schramm-Loewner evolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):697–724, 2009, 0711.1884. MR2571956 (2011g:60151)
- [Dub09b] J. Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009, 0712.3018. MR2525778 (2011d:60242)
- [Dup99a] B. Duplantier. Harmonic Measure Exponents for Two-Dimensional Percolation. *Physical Review Letters*, 82(3940), 1999, cond-mat/9901008.
- [Dup99b] B. Duplantier. Two-Dimensional Copolymers and Exact Conformal Multifractality. *Physical Review Letters*, 82(880), 1999, cond-mat/9812439.
- [Dup00] B. Duplantier. Conformally invariant fractals and potential theory. *Physical Review Letters*, 84(7):1363–1367, 2000, cond-mat/9908314.
- [Dup03] B. Duplantier. Higher conformal multifractality. *Journal of Statistical Physics*, 110(3–6):691–738, 2003, cond-mat/0207743.
- [Dup04] B. Duplantier. Conformal fractal geometry and boundary quantum gravity. In *Fractal Geometry and Applications: a Jubilee of Benôit Mandelbrot, Part 2, Proc. Sympos. Pure Math.*, 72, pages 365–482, Providence, RI, 2004. Amer. Math. Soc., math-ph/0303034.
- [HMP10] X. Hu, J. Miller, and Y. Peres. Thick points of the Gaussian free field. *Ann. Probab.*, 38(2):896–926, 2010, 0902.3842. MR2642894 (2011c:60117)
- [HS08] H. Hedenmalm and A. Sola. Spectral notions for conformal maps: a survey. *Comput. Methods Funct. Theory*, 8(1-2):447–474, 2008. MR2419488 (2009f:30002)
- [JVL11] F. Johansson Viklund and G. F. Lawler. Optimal Hölder exponent for the SLE path. *Duke Math. J.*, 159(3):351–383, 2011, 0904.1180. MR2831873
- [JVL12] F. Johansson Viklund and G. F. Lawler. Almost sure multifractal spectrum for the tip of an SLE curve. *Acta Math.*, 209(2):265–322, 2012, 0911.3983. MR3001607

- [Kra96] P. Kraetzer. Experimental bounds for the universal integral means spectrum of conformal maps. *Complex Variables Theory Appl.*, 31(4):305–309, 1996. MR1427159 (97m:30018)
- [Law96] G. Lawler. The dimension of the frontier of planar Brownian motion. *Electron. Comm. Probab.*, 1:no. 5, 29–47 (electronic), 1996. MR1386292 (97g:60110)
- [Law05] G. F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. MR2129588 (2006i:60003)
- [Law09] G. F. Lawler. Multifractal analysis of the reverse flow for the Schramm-Loewner evolution. In *Fractal geometry and stochastics IV*, volume 61 of *Progr. Probab.*, pages 73–107. Birkhäuser Verlag, Basel, 2009. MR2762674 (2012e:60209)
- [Lin08] J. R. Lind. Hölder regularity of the SLE trace. *Trans. Amer. Math. Soc.*, 360(7):3557–3578, 2008. MR2386236 (2009f:60048)
- [LSW01a] G. F. Lawler, O. Schramm, and W. Werner. The dimension of the planar Brownian frontier is $4/3$. *Math. Res. Lett.*, 8(4):401–411, 2001, math/0010165. MR1849257 (2003a:60127b)
- [LSW01b] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001, math/0003156. MR1879850 (2002m:60159a)
- [LSW01c] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. II. Plane exponents. *Acta Math.*, 187(2):275–308, 2001, math/9911084. MR1879851 (2002m:60159b)
- [LSW02] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. III. Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(1):109–123, 2002. MR1899232 (2003d:60163)
- [LSW03] G. Lawler, O. Schramm, and W. Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955 (electronic), 2003, math/0209343v2. MR1992830 (2004g:60130)
- [LSW04] G. F. Lawler, O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004, math/0112234. MR2044671 (2005f:82043)
- [LY13] I. Loutsenko and O. Yermolayeva. Average harmonic spectrum of the whole-plane SLE. *J. Stat. Mech. Theory Exp.*, (4):P04007, 17, 2013, 1203.2756. MR3077814
- [LY14] I. Loutsenko and O. Yermolayeva. New exact results in spectra of stochastic Loewner evolution. *J. Phys. A*, 47(16):165202, 15, 2014. MR3191679
- [Mak98] N. G. Makarov. Fine structure of harmonic measure. *Algebra i Analiz*, 10(2):1–62, 1998. MR1629379 (2000g:30018)
- [Mil10] J. Miller. Universality for SLE(4). *ArXiv e-prints*, October 2010, 1010.1356.
- [MP10] P. Mörters and Y. Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner. MR2604525 (2011i:60152)
- [MS13] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *ArXiv e-prints*, February 2013, 1302.4738.
- [MS16a] J. Miller and S. Sheffield. Imaginary Geometry I: Interacting SLEs. *Probability Theory and Related Fields*, to appear, 2016, 1201.1496.
- [MS16b] J. Miller and S. Sheffield. Imaginary geometry II: reversibility of $\text{SLE}_\kappa(\rho_1; \rho_2)$ for $\kappa \in (0, 4)$. *Annals of Probability*, to appear, 2016, 1201.1497.

- [MS16c] J. Miller and S. Sheffield. Imaginary geometry III: reversibility of SLE_κ for $\kappa \in (4, 8)$. *Annals of Mathematics*, to appear, 2016, 1201.1498.
- [MS16d] J. Miller and S. Sheffield. Quantum Loewner Evolution. *Duke Mathematics Journal*, to appear, 2016, 1312.5745.
- [MSW14] J. Miller, N. Sun, and D. B. Wilson. The Hausdorff dimension of the CLE gasket. *Ann. Probab.*, 42(4):1644–1665, 2014, 1403.6076. MR3262488
- [MW14] J. Miller and H. Wu. Intersections of SLE paths: the double and cut point dimension of SLE. *Probab. Theory Related Fields*, 2014, 1303.4725. To appear.
- [MWW14] J. Miller, S. S. Watson, and D. B. Wilson. Extreme nesting in the conformal loop ensemble. *ArXiv e-prints*, December 2014, 1401.0217.
- [Pom92] C. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. MR1217706 (95b:30008)
- [Pom97] C. Pommerenke. The integral means spectrum of univalent functions. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 237(Anal. Teor. Chisel i Teor. Funkts. 14):119–128, 229, 1997. MR1691287 (2000e:30031)
- [RS05] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005, math/0106036. MR2153402 (2006f:60093)
- [Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000, math/9904022. MR1776084 (2001m:60227)
- [She05] S. Sheffield. Local sets of the gaussian free field: slides and audio. www.fields.utoronto.ca/audio/05-06/percolationSLE/sheffield1, www.fields.utoronto.ca/audio/05-81,06/percolationSLE/sheffield2, www.fields.utoronto.ca/audio/05-06/percolationSLE/sheffield3, 2005.
- [She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007. MR2322706 (2008d:60120)
- [She16] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Annals of Probability*, to appear, 2016, 1012.4797.
- [Smi10] S. Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math. (2)*, 172(2):1435–1467, 2010, 0708.0039. MR2680496 (2011m:60302)
- [SS05] O. Schramm and S. Sheffield. Harmonic explorer and its convergence to SLE_4 . *Ann. Probab.*, 33(6):2127–2148, 2005, math/0310210. MR2184093 (2006i:60013)
- [SS09] O. Schramm and S. Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.*, 202(1):21–137, 2009, math/0605337. MR2486487 (2010f:60238)
- [SS13] O. Schramm and S. Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013, math/0605337. MR3101840
- [SW05] O. Schramm and D. B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669 (electronic), 2005, math/0505368. MR2188260 (2007e:82019)
- [Wer04] W. Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 107–195. Springer, Berlin, 2004, math/030335. MR2079672 (2005m:60020)
- [WW14] M. Wang and H. Wu. Level Lines of Gaussian Free Field I: Zero-Boundary GFF. *ArXiv e-prints*, December 2014, 1412.3839.

- [Zha08a] D. Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309–353, 2008, 0712.0332. MR2439609 (2010f:60239)
- [Zha08b] D. Zhan. Reversibility of chordal SLE. *Ann. Probab.*, 36(4):1472–1494, 2008. MR2435856 (2010a:60284)
- [Zha10] D. Zhan. Duality of chordal SLE, II. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(3):740–759, 2010, 0803.2223. MR2682265 (2011i:60155)